FURTHER IMPROVEMENTS IN WARING'S PROBLEM, IV: HIGHER POWERS

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1. Introduction

As usual we define $G(k)$ to be the least number s such that every sufficiently large natural number is the sum of, at most, s kth powers of natural numbers. In this paper we continue the program, initiated in Vaughan and Wooley [18] and extended in [16] and [17], of comprehensively developing the repeated efficient differencing process of Wooley [19]. Following Vaughan [13], our methods depend on upper bounds for the number, $S_s^{(k)}(P,R)$, of solutions of the diophantine equations

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k, \tag{1.1}$$

with $x_i, y_i \in \mathcal{A}(P,R)$, where throughout we write

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} \;:\; p \text{ prime, } p|n \text{ implies } p \leqslant R\}.$$

In Vaughan and Wooley [18] we established bounds for $G(k)$ when $5\leqslant k\leqslant 9$, and reported on preliminary bounds for $G(k)$ when $10\leqslant k\leqslant 15$. We now extend the latter calculations to bound $G(k)$ when $9\leqslant k\leqslant 20$, exploiting subsequent developments and making some further technical refinements.

Theorem 1.1. *When $9\leqslant k\leqslant 20$, one has $G(k)\leqslant H(k)$, where $H(k)$ is given in the following table.*

k	9	10	11	12	13	14	15	16	17	18	19	20
$H(k)$	50	59	67	76	84	92	100	109	117	125	134	142

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For comparison, Wooley [19, 22] has obtained the bounds $G(k)\leqslant 8k-18$ for $10\leqslant k\leqslant 17$, and $G(18)\leqslant 127$, $G(19)\leqslant 135$, $G(20)\leqslant 144$. Meanwhile, for smaller exponents one has the bounds

$$G(5)\leqslant 17,\quad G(6)\leqslant 24,\quad G(7)\leqslant 33,\quad G(8)\leqslant 42,\quad G(9)\leqslant 51,$$

by collecting together the conclusions of Vaughan and Wooley [16, 17, 18]. Indeed, we reported in [18] that preliminary calculations indicated the validity of the bounds

$$G(10)\leqslant 59,\quad G(11)\leqslant 67,\quad G(12)\leqslant 76,\quad G(13)\leqslant 84,\quad G(14)\leqslant 92,\quad G(15)\leqslant 100,$$

though we gave no details of these calculations. It is worth remarking that a number of authors have obtained estimates weaker than those established (or described) in Vaughan and Wooley [18] and Wooley [22] (see especially [8, 9, 10, 11]), following the publication of the papers [18] and [22]. However, the only improvement on the bounds contained in [18] and [22] known to the authors is a result of Meng [11], namely that $G(20)\leqslant 143$, and this is now superseded by Theorem 1.1.

It transpires that our estimates for the mean values $S_s^{(k)}(P,R)$ required in the proof of Theorem 1.1 are also of use in both localised and unlocalised estimates for the fractional part of αn^k.

Theorem 1.2. *Let $\alpha\in\mathbb{R}$ and $\varepsilon>0$. Then when $7\leqslant k\leqslant 20$, there is a real number $N(\varepsilon,k)$ with the property that whenever $N\geqslant N(\varepsilon,k)$, one has*

$$\min_{1\leqslant n\leqslant N}\|\alpha n^k\|\leqslant N^{\varepsilon-\sigma(k)},$$

where $\sigma(k)^{-1}=S(k)$, and $S(k)$ is given by the following table.

k	7	8	9	10	11	12	13
$S(k)$	57.23	69.66	82.08	94.62	107.27	119.78	132.34

k	14	15	16	17	18	19	20
$S(k)$	145.02	157.76	170.52	183.32	196.24	209.17	222.16

For comparison, Baker [1] shows that $\sigma(k)^{-1}=2^{k-1}$ is permissible in Theorem 1.2 for each k (following Danicic [5]), and describes how Vinogradov's methods yield sharper estimates for larger k. We note that Theorem 1.2 provides improvements on these exponents whenever $k\geqslant 7$. When k is large, meanwhile, the conclusion of Theorem 1.2 of Wooley [22] shows that $\sigma(k)^{-1}=k(\log k+O(\log\log k))$ is permissible.

Theorem 1.3. *Let $\alpha\in\mathbb{R}$ and $\varepsilon>0$. Then there are infinitely many natural numbers n with*

$$\|\alpha n^k\|\leqslant n^{\varepsilon-\tau(k)},$$

where $\tau(k)^{-1} = T(k)$*, and* $T(k)$ *is given by the following table.*

k	8	9	10	11	12	13	14
$T(k)$	57.72	67.25	76.71	86.18	95.46	104.77	114.02

k	15	16	17	18	19	20
$T(k)$	123.24	132.46	141.64	150.82	159.95	169.06

The conclusion of Theorem 1.3 may be compared with Corollary 2 to Theorem 1.1 of Wooley [20], which shows that the exponent $\tau(k)^{-1} = 9.028k$ is permissible for every k (improving on earlier work of Heath-Brown [7]). For smaller k, moreover, Heath-Brown [7] has shown that $\tau(k)^{-1} = 3 \cdot 2^{k-3}$ is permissible for $k \geqslant 6$. The conclusion of Theorem 1.3 improves on the latter for $k \geqslant 8$. For $k = 7$, meanwhile, our methods yield $\tau(7)^{-1} = 48.13$, which narrowly fails to surpass Heath-Brown's exponent $\tau(7)^{-1} = 48$. As noted by Heath-Brown [7], when α is algebraic, the method used to establish Theorem 1.3 shows, via an application of Roth's theorem, that the conclusion of Theorem 1.2 holds with $\sigma(k)$ replaced by $\tau(k)$.

Broadly speaking, our proof of Theorem 1.1 follows the pattern of Vaughan and Wooley [18]. We discuss the salient features of the underlying methods in §2 of this paper. The calculations involved in the proof are substantial, and thus one of the major challenges of this paper is the development of a strategy for handling the inherent complexity of our methods. There are three significant improvements on the methods of [18] of which we make use. Firstly, we employ the methods of Vaughan and Wooley [16], together with some refinements described in §5, in order to better handle the mean values of exponential sums over difference polynomials on the major arcs of our Hardy-Littlewood dissection. Such methods significantly enhance our estimates for mean values towards the end of the iteration process. Secondly, we make use of the new estimates for smooth Weyl sums contained in Wooley [22]. For larger k, these new estimates alone save several variables in the representations underlying our bounds for $G(k)$. Finally, in §3 of this paper, we establish new estimates for mean values of 2^l-th power moments of exponential sums over difference polynomials, establishing an important technical refinement of the corresponding estimates contained in [18]. Although these latter estimates are of significance only in the initial segment of the iteration process, they nonetheless lead to improvements in mean value estimates significant to the estimates recorded in Theorems 1.2 and 1.3. We remark that the highly technical estimates described in Wooley [21] offer the prospect of further refinements in the mean value estimates described herein. However, it would seem that for larger k, such improvements are not significant so far as bounds for $G(k)$ are concerned.

2. Preliminary observations

In order to put the work of the present paper in its proper setting, we first recall some of the notation and discussion of [18]. Throughout, k will denote an arbitrary integer exceeding 2, the letter s will denote a positive integer, and ε and η will denote sufficiently small positive numbers. We take P to be a large real number depending at most on k, s, ε and η. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k, s, ε and η. We make frequent use of vector notation for brevity. For example, $(c_1, \dots, c_t)$ is abbreviated to $\mathbf{c}$. Also, we write $e(\alpha)$ for $e^{2\pi i\alpha}$, and $[x]$ for the greatest integer not exceeding x.

In an effort to simplify our analysis, we adopt the following convention concerning the numbers ε and R. Whenever ε or R appear in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, there exists a positive number $\eta_0(\varepsilon, s, k)$ such that the statement holds whenever $R = P^\eta$, with $0 < \eta \leqslant \eta_0(\varepsilon, s, k)$. Note that the "value" of ε, and η_0, may change from statement to statement, and hence also the dependency of implicit constants on ε and η. Notice that since our iterative methods will involve only a finite number of statements (depending at most on k, s and ε), there is no danger of losing control of implicit constants through the successive changes implicit in our arguments. Finally, we use the symbol $\approx$ to indicate that constants and powers of R and P^ε are to be ignored.

For each $s \in \mathbb{N}$ we take $\phi_i = \phi_{i,s}$ $(i = 1, \dots, k)$ to be real numbers, with $0 \leqslant \phi_i \leqslant 1/k$, to be chosen later. We then take

$$P_j = 2^j P, \quad M_j = P^{\phi_j}, \quad H_j = P_j M_j^{-k}, \quad Q_j = P_j (M_1 \dots M_j)^{-1} \quad (0 \leqslant j \leqslant k),$$

and here, and throughout, the empty product is taken to be unity. We also write

$$\widetilde{H}_j = \prod_{i=1}^{j} H_i \quad \text{and} \quad \widetilde{M}_j = \prod_{i=1}^{j} M_i R.$$

We define the modified forward difference operator, Δ_1^*, by

$$\Delta_1^* \left(f(x); h; m\right) = m^{-k} \left(f(x + hm^k) - f(x)\right),$$

and define Δ_j^* recursively by

$$\Delta_{j+1}^* \left(f(x); h_1, \dots, h_{j+1};\ m_1, \dots, m_{j+1}\right) = \Delta_1^* \left(\Delta_j^* \left(f(x); h_1, \dots, h_j; m_1, \dots, m_j\right); h_{j+1}; m_{j+1}\right).$$

We also adopt the convention that $\Delta_0^* \left(f(x); h; m\right) = f(x)$.

For $0 \leqslant j \leqslant k$ let

$$\Psi_j = \Psi_j(z; h_1, \dots, h_j; m_1, \dots, m_j) = \Delta_j^* \left(f(z); 2h_1, \dots, 2h_j; m_1, \dots, m_j\right),$$

where $f(z) = (z - h_1 m_1^k - \dots - h_j m_j^k)^k$.

We write

$$f_j(\alpha) = \sum_{x \in \mathcal{A}(Q_j, R)} e(\alpha x^k), \quad f_j^+(\alpha) = \sum_{\substack{x \in \mathcal{A}(Q_j, R) \\ x > \frac{1}{2} Q_j R^{-j}}} e(\alpha x^k)$$

and

$$g_j(\alpha) = \sum_{\frac{1}{2}Q_jR^{-j} < x \leqslant Q_j} e(\alpha x^k),$$

and write also

$$F_j(\alpha) = \sum_{z,\mathbf{h},\mathbf{m}} e\left(\alpha \Psi_j(z;\mathbf{h};\mathbf{m})\right),$$

where the summation is over $z, \mathbf{h}, \mathbf{m}$ with

$$1 \leqslant z \leqslant P_j, \quad M_i < m_i \leqslant M_i R, \quad m_i \in \mathcal{A}(P,R), \quad 1 \leqslant h_i \leqslant 2^{j-i} H_i, \tag{2.1}$$

for $1 \leqslant i \leqslant j$. We define $S_s^{(k)}(P,R)$ as in the introduction. Suppose that the real numbers $\lambda_s^{(k)}$ $(1 \leqslant s < \infty)$ have the property that

$$S_s^{(k)}(P,R) \ll P^{\lambda_s^{(k)}+\varepsilon}.$$

Then we say that the $\lambda_s^{(k)}$ are *permissible* exponents. Such numbers certainly exist, since we may trivially take $\lambda_s^{(k)} = 2s$. Then for each s, we define the quantity $\Delta_s^{(k)}$ by

$$\lambda_s^{(k)} = 2s - k + \Delta_s^{(k)}.$$

When $\lambda_s^{(k)}$ is a permissible exponent, we say that $\Delta_s^{(k)}$ is an *admissible* exponent. When no confusion is possible, we suppress the superscript k.

The efficient differencing process which underlies our arguments is implicit in the following lemmata.

Lemma 2.1. *We have*

$$\int_0^1 \left|F_0(\alpha)^2 f_0(\alpha)^{2s}\right| d\alpha \ll P^\varepsilon M_1^{2s-1}\left(PM_1Q_1^{\lambda_s} + \int_0^1 \left|F_1(\alpha) f_1(\alpha)^{2s}\right| d\alpha\right). \tag{2.2}$$

Further, the inequality (2.2) *holds also when $f_i(\alpha)$ is replaced by $f_i^+(\alpha)$ for $i = 0, 1$.*

Proof. The inequality (2.2) is immediate from Lemma 2.1 of [18]. Meanwhile, a consideration of the intermediate underlying diophantine equations reveals that the replacement of the exponential sums $f_i(\alpha)$ by $f_i^+(\alpha)$ $(i = 0, 1)$ is easily accomodated within the argument of the proofs of Lemmata 2.2 and 2.3 of Wooley [19], and thus the second conclusion of the lemma also follows with minimal effort.

Following [18], we abbreviate an inequality of the form (2.2) symbolically by

$$F_0^2 f_0^{2s} \longmapsto F_1 f_1^{2s},$$

with a similar convention when f_i is replaced by f_i^+ $(i = 0, 1)$.

Lemma 2.2. *Whenever* $0 < t < 2s$ *and* $1 \leqslant j \leqslant k-1$*, we have*

$$\int_0^1 \left|F_j(\alpha)f_j(\alpha)^{2s}\right| d\alpha \ll P^\varepsilon (Q_j^{\lambda_t})^{1/2} (\widetilde{H}_j \widetilde{M}_j M_{j+1}^{4s-2t-1} T_{j+1})^{1/2}, \tag{2.3}$$

where

$$T_{j+1} = P\widetilde{H}_j \widetilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} + \int_0^1 \left|F_{j+1}(\alpha) f_{j+1}(\alpha)^{4s-2t}\right| d\alpha. \tag{2.4}$$

Further, the inequality (2.3) *holds also when* $f_i(\alpha)$ *is replaced by* $f_i^+(\alpha)$ *for* $i = j, j+1$.

Proof. The first conclusion of the lemma is immediate from Lemma 2.2 of [18], on correcting a typographic error in the statement of the latter. The second conclusion of the lemma follows as in the proof of Lemma 2.2 of [18], on making use of Lemmata 2.3 and 3.1 of Wooley [19], the replacement of the exponential sum $f_i(\alpha)$ by $f_i^+(\alpha)$ $(i = j, j+1)$ leading to minor cosmetic changes only.

Lemma 2.3. *Whenever* $0 < t < 2s$ *and* $1 \leqslant j \leqslant k-1$*, we have*

$$\int_0^1 \left|F_j(\alpha)f_j^+(\alpha)^{2s}\right| d\alpha \ll P^\varepsilon (Q_j^{\lambda_t})^{1/2} (\widetilde{H}_j \widetilde{M}_j M_{j+1}^{4s-2t-1} \tilde{T}_{j+1})^{1/2}, \tag{2.5}$$

where

$$\tilde{T}_{j+1} = P\widetilde{H}_j \widetilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} + \int_0^1 \left|F_{j+1}(\alpha) g_{j+1}(\alpha)^2 f_{j+1}(\alpha)^{4s-2t-2}\right| d\alpha. \tag{2.6}$$

Proof. The proof is based on the use of Lemmata 2.3 and 3.1 of Wooley [19], in a manner similar to the proof of Lemma 2.2 of Vaughan and Wooley [18]. By applying Schwarz's inequality as in the proof of Lemma 2.2 of [18], we find that

$$\int_0^1 |F_j(\alpha)f_j^+(\alpha)^{2s}| d\alpha \ll \Big(\int_0^1 |f_j^+(\alpha)|^{2t} d\alpha\Big)^{1/2} \Big(\int_0^1 |F_j(\alpha)^2 f_j^+(\alpha)^{4s-2t}| d\alpha\Big)^{1/2}. \tag{2.7}$$

But on considering the underlying diophantine equations,

$$\int_0^1 |f_j^+(\alpha)|^{2t} d\alpha \leqslant \int_0^1 |f_j(\alpha)|^{2t} d\alpha \ll Q_j^{\lambda_t+\varepsilon}, \tag{2.8}$$

and by the argument of the proof of Lemmata 2.3 and 3.1 of [19], again noting that the replacement of $f_i(\alpha)$ by $f_i^+(\alpha)$ is easily accomodated, one finds that

$$\int_0^1 |F_j(\alpha)^2 f_j^+(\alpha)^{4s-2t}| d\alpha \ll P^\varepsilon \widetilde{H}_j \widetilde{M}_j M_{j+1}^{4s-2t-1} \left(P\widetilde{H}_j \widetilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} + T_{j+1}^*\right), \tag{2.9}$$

where T^*_{j+1} denotes the number of solutions of the diophantine equation

$$\Psi_{j+1}(z;\mathbf{h};\mathbf{m}) + \sum_{i=1}^{2s-t}(x_{2i-1}^k - x_{2i}^k) = 0, \tag{2.10}$$

with $z, \mathbf{h}, \mathbf{m}$ satisfying (2.1) for $1 \leqslant i \leqslant j+1$, and with $x_i \in \mathcal{A}(Q_{j+1}, R)$ and $x_i > \frac{1}{2}Q_{j+1}R^{-j-1}$ for $1 \leqslant i \leqslant 4s-2t$. Here we note that our summation conditions differ from those of (3.3) of [19] only by virtue of the notation defined above, and the latter condition on the x_i.

Let T^+_{j+1} denote the number of solutions of the diophantine equation (2.10) with z, h_i, m_i satisfying (2.1) for $1 \leqslant i \leqslant j+1$, and with

$$\tfrac{1}{2}Q_{j+1}R^{-j-1} < x_1, x_2 \leqslant Q_{j+1} \quad \text{and} \quad x_i \in \mathcal{A}(Q_{j+1}, R) \quad (3 \leqslant i \leqslant 4s-2t).$$

Then it is evident that $T^*_{j+1} \leqslant T^+_{j+1}$, and moreover, on considering the underlying diophantine equations,

$$T^+_{j+1} = \int_0^1 F_{j+1}(\alpha)|g_{j+1}(\alpha)^2 f_{j+1}(\alpha)^{4s-2t-2}|\,d\alpha \leqslant \tilde{T}_{j+1}.$$

The conclusion of the lemma therefore follows by combining (2.7)-(2.9).

We abbreviate inequalities of the form (2.3) and (2.5) symbolically by

$$\begin{array}{ccc} F_j f_j^{2s} & \longrightarrow & F_{j+1} f_{j+1}^{4s-2t} \\ & & \downarrow \\ & & f_j^{2t} \end{array}$$

and

$$\begin{array}{ccc} F_j {f_j^+}^{2s} & \longrightarrow & F_{j+1} g_{j+1}^2 f_{j+1}^{4s-2t-2} \\ & & \downarrow \\ & & f_j^{2t} \end{array}$$

respectively.

The integrals on the right hand side of (2.2), (2.4) and (2.6) may be estimated in two ways other than simply repeating the efficient differencing process.

Firstly, we may apply Hölder's inequality in the form

$$\int_0^1 \left|F_j(\alpha) f_j(\alpha)^{2s}\right| d\alpha \ll I_l^a I_{l+1}^b U_v^c U_w^d \tag{2.11}$$

where

$$I_m = \int_0^1 |F_j(\alpha)|^{2^m}\, d\alpha \quad (m = l, l+1)$$

and

$$U_u = \int_0^1 |f_j(\alpha)|^{2u}\, d\alpha \quad (u = v, w),$$

in which l, v and w are non-negative integers, and a, b, c, d are non-negative real numbers with

$$a + b + c + d = 1, \quad 2^l a + 2^{l+1} b = 1, \quad vc + wd = s.$$

The 2^m-th power mean values of F_j may be estimated in terms of the number of solutions of certain diophantine equations, as we describe below. Also, we have $U_v \ll Q_j^{\lambda_v+\varepsilon}$ and $U_w \ll Q_j^{\lambda_w+\varepsilon}$. We abbreviate an inequality of the shape (2.11) symbolically by

$$F_j f_j^{2s} \Longrightarrow (F_j^{2^l})^a (F_j^{2^{l+1}})^b (f_j^{2v})^c (f_j^{2w})^d.$$

We discuss such inequalities further in §4 below.

Secondly, we may apply the Hardy-Littlewood method along the lines of Vaughan and Wooley [16]. We then abbreviate the resulting inequality symbolically in the form

$$F_j f_j^{2s} \Longrightarrow (F_j)(f_j^{2s})$$

or

$$F_j g_j^2 f_j^{2s-2} \Longrightarrow (F_j)(g_j^2 f_j^{2s-2}).$$

We discuss the material from [16] required in this paper in §5 below.

By considering the underlying diophantine equations, we have

$$S_{s+1}(P, R) \leqslant \int_0^1 \left| F_0(\alpha)^2 f_0(\alpha)^{2s} \right| d\alpha.$$

Also, on writing

$$H(\alpha; Q) = \sum_{1 \leqslant x \leqslant Q} e(\alpha x^k) \quad \text{and} \quad h(\alpha; Q) = \sum_{\substack{x \in \mathcal{A}(Q,R) \\ x > Q/2}} e(\alpha x^k),$$

it follows from a consideration of the underlying diophantine equations and Hölder's inequality that

$$\begin{aligned} S_{s+1}(P, R) &\leqslant \int_0^1 \Big| \sum_{\substack{i=0 \\ 2^i \leqslant \sqrt{P}}}^{\infty} h(\alpha; 2^{-i}P) + H(\alpha; \sqrt{P}) \Big|^{2s+2} d\alpha \\ &\ll P^{s+1} + (\log P)^{2s+2} \max_{\substack{0 \leqslant i < \infty \\ 2^i \leqslant \sqrt{P}}} \int_0^1 |h(\alpha; 2^{-i}P)|^{2s+2} d\alpha \\ &\ll P^{s+1} + P^{\varepsilon} \max_{\substack{0 \leqslant i < \infty \\ 2^i \leqslant \sqrt{P}}} \int_0^1 |H(\alpha; 2^{-i}P)^2 h(\alpha; 2^{-i}P)^{2s}| d\alpha. \end{aligned}$$

Since the last integral has the shape

$$\int_0^1 |F_0(\alpha)^2 f_0^+(\alpha)^{2s}|d\alpha,$$

it follows in either case that we may use a sequence of connected inequalities (in the obvious sense) to bound $S_s(Q,R)$ in terms of $S_t(Q',R)$ $(t=1,2,\dots)$. By optimising parameters one obtains in this way a permissible exponent λ_{s+1} for which

$$S_{s+1}(P,R) \ll P^{\lambda_{s+1}+\varepsilon}.$$

The use of such inequalities within an iterative process is discussed in detail in §2 of [18]. While we avoid detailed discussion of such issues, we will indicate the manner in which optimal parameters are to be found. Notice that whenever the methods of this paper establish that λ_{s+1} is a permissible exponent, then on considering the underlying diophantine equations, it is apparent that they also establish the upper bound

$$\int_0^1 |F_0(\alpha)^2 f_0^+(\alpha)^{2s}|d\alpha \ll P^{\lambda_{s+1}+\varepsilon}, \tag{2.12}$$

since our starting point in deriving such a permissible exponent is an application of Lemma 2.1. Either the mean value on the left hand side of (2.12) occurs explicitly in the latter application, or else a similar expression in which f_0^+ is replaced by f_0, and of course a consideration of the underlying diophantine equations readily confirms that this last mean value majorises the former.

Finally, having established estimates for the mean values $S_s(P,R)$, one must still employ these bounds within the proofs of Theorems 1.1, 1.2 and 1.3. We discuss the latter details in §§6 and 7. The calculation of the exponents $\lambda_s^{(k)}$, and subsequent computation of $G(k)$, $\tau(k)$, $\sigma(k)$ we defer to §§8 to 23.

3. Estimates for the number of solutions of auxiliary equations

In this section we explore some technical refinements of the methods of §3 of [18] concerning the moments of the exponential sum $F_j(\alpha)$. Before proceeding further we require some notation. Let $R_j^{(s)}(P;\boldsymbol{\phi})$ denote the number of solutions of the diophantine equation

$$\sum_{i=1}^{s} \Psi_j\left(z_i;\mathbf{h}^{(i)};\mathbf{m}^{(i)}\right) = \sum_{i=1}^{s} \Psi_j\left(w_i;\mathbf{g}^{(i)};\mathbf{n}^{(i)}\right), \tag{3.1}$$

with

$$1 \leqslant z_i, w_i \leqslant P_j, \quad 1 \leqslant h_t^{(i)}, g_t^{(i)} \leqslant 2^{j-t} H_t, \tag{3.2}$$

$$M_t < m_t^{(i)}, n_t^{(i)} \leqslant M_t R, \quad m_t^{(i)}, n_t^{(i)} \in \mathcal{A}(P,R), \tag{3.3}$$

for $1\leqslant t\leqslant j$ and $1\leqslant i\leqslant s$. Also, we put $J=\left[\frac{1}{2}(k-j)\right]$, and define $K_j(P;\boldsymbol{\phi})$ to be the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{j} h_i^{2r}(m_i^{2rk}-n_i^{2rk})=0 \quad (1\leqslant r\leqslant J), \tag{3.4}$$

with $\mathbf{h}$, $\mathbf{m}$ and $\mathbf{n}$ in the ranges defined by (3.2) and (3.3).

It transpires that the estimates for $R_j^{(s)}=R_j^{(s)}(P;\boldsymbol{\phi})$ that are presently attainable are of interest only when s is a small power of 2. Moreover, one may bound $R_j^{(2^l)}$ in terms of $R_j^{(2^{l-1})}$ and $R_{j+l}^{(1)}$ as follows.

Lemma 3.1. *When $1\leqslant l\leqslant k-2$ and $1\leqslant j\leqslant k-l-1$, one has*

$$\begin{aligned} R_j^{(2^l)}(P;\phi_1,\dots,\phi_j) \ll{}& P^{2^l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l}R_j^{(2^{l-1})}(P;\phi_1,\dots,\phi_j)\\ &+P^{2^{l+1}-2l-2}(\widetilde{H}_j\widetilde{M}_j)^{2^{l+1}-2}R_{j+l}^{(1)}(P;\phi_1,\dots,\phi_j,0,\dots,0).\end{aligned}$$

Proof. This is a natural development of the proof of Lemma 3.1 of [18]. On considering the underlying diophantine equations, it follows from (3.1) that

$$R_j^{(2^l)}(P;\boldsymbol{\phi})=\int_0^1 |F_j(\alpha)|^{2^{l+1}}\,d\alpha.$$

Write

$$\Psi_{j,l}(z;\mathbf{h};\mathbf{m};\mathbf{u})=\Delta_l^*(\Psi_j(z;\mathbf{h};\mathbf{m});\mathbf{u};1,\dots,1),$$

in which $\mathbf{u}=(u_1,\dots,u_l)$. Then by applying standard Weyl differencing (see, for example, Lemma 2.3 of Vaughan [15]), an application of Hölder's inequality reveals that

$$|F_j(\alpha)|^{2^l}\ll P^{2^l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l}+P^{2^l-l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l-1}|G(\alpha)|,$$

where

$$G(\alpha)=\sum_{\mathbf{h},\mathbf{m}}\sum_{1\leqslant u_1\leqslant P_j}\cdots\sum_{1\leqslant u_l\leqslant P_j}\sum_{\substack{1\leqslant z\leqslant P_j-u_1-\dots-u_l\\ z\in I(\mathbf{u})}} e(\alpha\Psi_{j,l}(z;\mathbf{h};\mathbf{m};\mathbf{u})),$$

and here $I(\mathbf{u})$ denotes an interval depending only on $\mathbf{u}$, and the summation over $\mathbf{h}$ and $\mathbf{m}$ is over the ranges given in (3.2) and (3.3). Thus we deduce that

$$\begin{aligned} R_j^{(2^l)}(P;\boldsymbol{\phi}) \ll{}& P^{2^l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l}\int_0^1 |F_j(\alpha)|^{2^l}\,d\alpha\\ &+P^{2^l-l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l-1}\int_0^1 |G(\alpha)F_j(\alpha)^{2^l}|\,d\alpha.\end{aligned}$$

An application of Schwarz's inequality, combined with a consideration of the underlying diophantine equations, therefore reveals that

$$\begin{aligned} R_j^{(2^l)}(P;\phi) \ll & P^{2^l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l} R_j^{(2^{l-1})}(P;\phi) \\ & + P^{2^l-l-1}(\widetilde{H}_j\widetilde{M}_j)^{2^l-1}\left(R_j^{(2^l)}(P;\phi)S\right)^{1/2}, \end{aligned}$$

where S denotes the number of solutions of the equation

$$\Psi_{j,l}(z;\mathbf{h};\mathbf{m};\mathbf{u}) = \Psi_{j,l}(w;\mathbf{g};\mathbf{n};\mathbf{v}),$$

with the variables $\mathbf{h},\mathbf{g},\mathbf{m},\mathbf{n}$ in the ranges defined by (3.2) and (3.3), and with

$$1\leqslant u_i, v_i\leqslant P_j \quad (1\leqslant i\leqslant l),$$

$$1\leqslant z\leqslant P_j - u_1 - \cdots - u_l, \quad 1\leqslant w\leqslant P_j - v_1 - \cdots - v_l.$$

But we have

$$\begin{aligned} & 2^k\Delta_l^*(\Psi_j(z;\mathbf{h};\mathbf{m});\mathbf{u};1,\dots,1) \\ & \quad = \Delta_{j+l}^*((2z-2h_1m_1^k-\cdots-2h_jm_j^k)^k;4\mathbf{h},2\mathbf{u};\mathbf{m},1,\dots,1) \\ & \quad = \Psi_{j+l}(2z+u_1+\cdots+u_l;2\mathbf{h},\mathbf{u};\mathbf{m},1,\dots,1). \end{aligned}$$

The desired conclusion therefore follows on noting that

$$2z+u_1+\cdots+u_l < 2P_j\leqslant P_{j+l}.$$

The methods of §3 of [18] provide a bound of the shape

$$R_j^{(1)}(P;\phi) \ll P^{1+\varepsilon}K_j(P;\phi)$$

when $j=1$, and for $1\leqslant j\leqslant k-2$ in circumstances in which $k-j$ is odd, and also when $k-j=2$ or 4, but in all other circumstances the bounds obtained are somewhat unsatisfactory. Our primary aim in this section is to treat as many of the cases in which $k-j$ is even as is practicable. We handle the latter cases by making use of an estimate for the number of integral points on certain affine plane curves due to Bombieri and Pila [2] (this idea was mentioned to us in a conversation by Professor E. Bombieri in early 1991).

Lemma 3.2. *Let $\mathcal{C}$ be the curve defined by the equation $F(x,y)=0$, where $F(x,y)\in\mathbb{R}[x,y]$ is an absolutely irreducible polynomial of degree $d\geqslant 2$. Also, let $N\geqslant\exp(d^6)$. Then the number of integral points on $\mathcal{C}$, and inside a square $[0,N]\times[0,N]$, does not exceed*

$$N^{1/d}\exp(12(d\log N\log\log N)^{1/2}).$$

Proof. This is Theorem 5 of Bombieri and Pila [2].

Before announcing the new estimate at the heart of this discussion, we recall from §3 of [18] that for each j and k one has

$$\Psi_j(z;\mathbf{h};\mathbf{m})=k!2^jh_1\dots h_j\sum_{u\geqslant 0}\sum_{v_1\geqslant 0}\cdots\sum_{v_j\geqslant 0}\frac{z^u(h_1m_1^k)^{2v_1}\dots(h_jm_j^k)^{2v_j}}{u!(2v_1+1)!\dots(2v_j+1)!},$$

where the summation is subject to the condition $u+2v_1+\dots+2v_j=k-j$. Consequently, when $k-j$ is even, one has that

$$\Psi_j(z;\mathbf{h};\mathbf{m})=h_1\dots h_j\sum_{r=0}^{J}c_r(h_1m_1^k,\dots,h_jm_j^k)z^{2r}, \tag{3.5}$$

where the $c_r(\xi_1,\dots,\xi_j)\in\mathbb{Z}[\boldsymbol{\xi}]$ are polynomials with positive coefficients which are symmetric in $\xi_1^2,\dots,\xi_j^2$ of degree $J-r$ $(0\leqslant r\leqslant J)$.

Lemma 3.3. *Suppose that* $1\leqslant j\leqslant k-6$ *and* $k-j$ *is even. Then*

$$R_j^{(1)}(P;\boldsymbol{\phi})\ll P^{1+\varepsilon}K_j(P;\boldsymbol{\phi})+P^{2/3+\varepsilon}\widetilde{H}_j\widetilde{M}_j^2.$$

Proof. Observe first that in view of (3.5), the polynomial $\Psi_j(z;\mathbf{h};\mathbf{m})$ is divisible by $h_1\dots h_j$. The argument of the proof of Lemma 3.2 of [18] therefore shows that

$$R_j^{(1)}(P;\boldsymbol{\phi})\ll P^{\varepsilon}R_j^*(P;\boldsymbol{\phi}), \tag{3.6}$$

where now we write $R_j^*(P;\boldsymbol{\phi})$ for the number of solutions of the equation

$$\Psi_j(z;\mathbf{h};\mathbf{m})=\Psi_j(w;\mathbf{h};\mathbf{n}), \tag{3.7}$$

with $z,w,\mathbf{h},\mathbf{m},\mathbf{n}$ satisfying

$$1\leqslant z,w\leqslant P_j,\quad 1\leqslant h_i\leqslant 2^{j-i}H_i,\quad m_i,n_i\in\mathcal{A}(P,R)\cap(M_i,M_iR]\quad(1\leqslant i\leqslant j). \tag{3.8}$$

We divide our argument into cases. Let R_0 denote the number of solutions of the equation (3.7) counted by $R_j^*(P;\boldsymbol{\phi})$ in which

$$c_0(h_1m_1^k,\dots,h_jm_j^k)=c_0(h_1n_1^k,\dots,h_jn_j^k), \tag{3.9}$$

and let R_1 denote the corresponding number of solutions in which (3.9) does not hold. Then one has

$$R_j^*(P;\boldsymbol{\phi})=R_0+R_1. \tag{3.10}$$

Observe first that if $z,w,\mathbf{h},\mathbf{m},\mathbf{n}$ is any solution of (3.7) counted by R_0, then it follows from (3.5) and (3.9) that

$$z^2\sum_{r=1}^{J}c_r(h_1m_1^k,\dots,h_jm_j^k)z^{2r-2}=w^2\sum_{r=1}^{J}c_r(h_1n_1^k,\dots,h_jn_j^k)w^{2r-2}. \tag{3.11}$$

When t and x are positive integers with $1\leqslant x\leqslant P_j$, and $\mathbf{h}$ satisfies (3.8), denote by $r(n;x,t;\mathbf{h})$ the number of solutions of the simultaneous diophantine equations

$$x^2\sum_{r=1}^J c_r(h_1m_1^k,\ldots,h_jm_j^k)x^{2r-2}=n,$$
$$c_0(h_1m_1^k,\ldots,h_jm_j^k)=t,$$

with $\mathbf{m}$ satisfying (3.8). Then it follows from (3.11) via Cauchy's inequality and an elementary estimate for the divisor function that

$$\begin{aligned}R_0&=\sum_{n=1}^\infty\sum_{t=1}^\infty\sum_{\mathbf{h}}\Big(\sum_{\substack{x^2|n\\ 1\leqslant x\leqslant P_j}} r(n;x,t;\mathbf{h})\Big)^2\\ &\ll P^\varepsilon\sum_{n=1}^\infty\sum_{t=1}^\infty\sum_{\mathbf{h}}\sum_{\substack{x^2|n\\ 1\leqslant x\leqslant P_j}} r(n;x,t;\mathbf{h})^2,\end{aligned}$$

where the summation over $\mathbf{h}$ is subject to (3.8). Thus it follows that

$$R_0\ll P^\varepsilon R_0^*, \tag{3.12}$$

where R_0^* denotes the number of solutions of the simultaneous diophantine equations

$$\sum_{r=1}^J\big(c_r(h_1m_1^k,\ldots,h_jm_j^k)-c_r(h_1n_1^k,\ldots,h_jn_j^k)\big)\,z^{2r-2}=0, \tag{3.13}$$
$$c_0(h_1m_1^k,\ldots,h_jm_j^k)-c_0(h_1n_1^k,\ldots,h_jn_j^k)=0, \tag{3.14}$$

with $z,\mathbf{h},\mathbf{m},\mathbf{n}$ satisfying (3.8).

Consider next the solutions of the system (3.13), (3.14) in which

$$c_r(h_1m_1^k,\ldots,h_jm_j^k)\neq c_r(h_1n_1^k,\ldots,h_jn_j^k)$$

for some r with $1\leqslant r\leqslant J$. We may assign $\mathbf{h}$, $\mathbf{m}$ and $\mathbf{n}$ in $O(\widetilde{H}_j\widetilde{M}_j^2)$ ways. Fixing any one such choice, it follows that z is determined by the non-trivial polynomial equation (3.13). Then there are $O(1)$ possible choices for z, and consequently the total number of solutions counted by R_0^* of this type is $O(\widetilde{H}_j\widetilde{M}_j^2)$. The remaining solutions $z,\mathbf{h},\mathbf{m},\mathbf{n}$ of the system (3.13), (3.14) counted by R_0^* satisfy the system

$$c_r(h_1m_1^k,\ldots,h_jm_j^k)=c_r(h_1n_1^k,\ldots,h_jn_j^k)\quad(0\leqslant r\leqslant J), \tag{3.15}$$

with the variable z unconstrained. On recalling that the polynomials $c_r(\boldsymbol{\xi})$ have positive coefficients and are symmetric in $\xi_1^2,\ldots,\xi_j^2$ of degree $J-r$ $(0\leqslant r\leqslant J)$, we find from (3.15) that the equations (3.4) are satisfied. Consequently, the number of possible choices for $\mathbf{h}, \mathbf{m}, \mathbf{n}$ is at most $K_j(P; \boldsymbol{\phi})$. We may therefore conclude that

$$R_0^* \ll \widetilde{H}_j \widetilde{M}_j^2 + PK_j(P; \boldsymbol{\phi}). \tag{3.16}$$

On noting that the diagonal solutions of (3.4) alone yield $\gg \widetilde{H}_j \widetilde{M}_j$ solutions, and recalling that our hypotheses on $\boldsymbol{\phi}$ dictate that $\widetilde{M}_j \ll P$, we find from (3.12) and (3.16) that

$$R_0 \ll P^{1+\varepsilon} K_j(P; \boldsymbol{\phi}). \tag{3.17}$$

Let $\mathbf{h}, \mathbf{m}, \mathbf{n}$ be any one of the $O(\widetilde{H}_j \widetilde{M}_j^2)$ possible choices satisfying (3.8) for which the equation (3.9) does not hold. Write

$$F(x, y) = \sum_{r=0}^{J} \left(c_r(h_1 m_1^k, \dots, h_j m_j^k) x^r - c_r(h_1 n_1^k, \dots, h_j n_j^k) y^r \right).$$

Then it follows that for this fixed choice of $\mathbf{h}, \mathbf{m}, \mathbf{n}$, the choices of z and w to be counted by R_1 satisfy the equation

$$F(z^2, w^2) = 0, \tag{3.18}$$

with $1 \leqslant z, w \leqslant P_j$, and moreover the constant term in (3.18) is non-zero. Suppose first that the polynomial $F(x, y)$ is absolutely irreducible. Then it follows from Lemma 3.2 that the number of possible choices for x and y with $1 \leqslant x, y \leqslant P_j^2$, satisfying the equation $F(x, y) = 0$, is $O(P^{2/J+\varepsilon})$. Hence the number of solutions of the equation (3.18) with $1 \leqslant z, w \leqslant P_j$ is similarly $O(P^{2/J+\varepsilon})$.

If, on the other hand, the polynomial $F(x, y)$ is not absolutely irreducible, then one may write $F(x, y)$ as a product of absolutely irreducible factors, say

$$F(x, y) = \prod_{i=1}^{l} g_i(x, y) \prod_{e=1}^{m} h_e(x, y), \tag{3.19}$$

where $l + m \geqslant 2$, and where $g_i(x, y) \in \mathbb{R}[x, y]$ $(1 \leqslant i \leqslant l)$, and

$$h_e(x, y) = u_e(x, y) + v_e(x, y)\sqrt{-1} \quad (1 \leqslant e \leqslant m),$$

with $u_e, v_e \in \mathbb{R}[x, y]$. We may suppose, moreover, that for each e the polynomials u_e and v_e have no non-trivial polynomial common divisor over $\mathbb{C}[x, y]$. It therefore follows from Bezout's Theorem that the number of solutions of the simultaneous equations $u_e(x, y) = v_e(x, y) = 0$ is bounded above by J^2. By considering real and imaginary components, therefore, the number of integral solutions of the equation $h_e(x, y) = 0$ is also bounded above by J^2. If the degree of $g_i(x, y)$ exceeds 2 for any i, then the absolute irreducibility of $g_i(x, y)$ ensures, via Lemma 3.2, that the number of integral solutions of the equation $g_i(x, y) = 0$, with $(x, y) \in [0, P_j^2]^2 \cap$

$\mathbb{Z}^2$, is $O(P^{2/3+\varepsilon})$. Consequently, it follows from the conclusion of the previous paragraph together with (3.19) that the number of solutions of the equation (3.18) with $1\leqslant z,w\leqslant P$ is $O(P^{2/3+\varepsilon})$, except possibly when $F(x,y)$ factorises in the form (3.19) with one at least of the g_i having degree one or two.

Suppose then that for some i with $1\leqslant i\leqslant l$, the polynomial $g_i(x,y)$ is quadratic or linear. If $g_i(x,y)$ is not some constant multiple of a $\mathbb{Q}$-rational polynomial, then since $g_i(x,y)$ is necessarily a constant multiple of a polynomial with algebraic coefficients, we deduce that the number of integral solutions of the equation $g_i(x,y)=0$ is at most $O(1)$. For we may remove the constant factor and consider components with respect to some basis for the field extension containing the coefficients of $g_i(x,y)$. Then since $g_i(x,y)$ is not a constant multiple of a $\mathbb{Q}$-rational polynomial, it follows that the integral zeros of the polynomial $g_i(x,y)$ necessarily satisfy at least two linearly independent $\mathbb{Q}$-rational equations, whence the desired conclusion follows from Bezout's Theorem.

Suppose next that for some i with $1\leqslant i\leqslant l$, the polynomial $g_i(x,y)$ is quadratic or linear, and has integral coefficients, as we may. Observe that the homogeneous part of $F(x,y)$ of maximal degree has the shape $\alpha(x^J-y^J)$, for a certain positive integer α. Thus any quadratic factor of $F(x,y)$ must have homogeneous part of the shape $\alpha_1\phi(x,y)$, where α_1 is rational and $\phi(x,y)$ is a divisor of x^J-y^J with rational coefficients. By cyclotomy, the only possibilities for $\phi(x,y)$ are therefore $x^2\pm y^2$ and $x^2\pm xy+y^2$. Further, similarly, any linear factor of $F(x,y)$ must have homogeneous part of the shape $\alpha_2(x\pm y)$, where α_2 is a rational number. In the latter case one has that $g_i(x,y)$ has the shape $a(x\pm y)+c$, for a certain non-zero integer a, and an integer c. Moreover, since the constant term in (3.18) is non-zero, one has $c\neq 0$. But then the number of solutions z,w of the equation (3.18), with $1\leqslant z,w\leqslant P_j$, which arise from the vanishing of the factor g_i, is bounded above by the number of solutions of the equation

$$a(z^2\pm w^2)+c=0, \tag{3.20}$$

with $1\leqslant z,w\leqslant P_j$. But standard estimates for the number of solutions of such quadratic equations (see, for example, Estermann [6] or Lemma 3.5 of [18]) reveal that the number of solutions of the equation (3.20) counted by R_1 is at most $O(P^{\varepsilon})$. If, on the other hand, the polynomial $g_i(x,y)$ is in fact a quadratic polynomial with rational coefficients, then in view of our earlier observation we may make a non-singular rational change of variables, $x=u+C_1$, $y=v+C_2$, so that the polynomial $g_i(x,y)$ takes the shape $a\phi(u,v)+c$ with a and c integers, and with $\phi(u,v)$ as above. The absolute irreducibility of $g_i(x,y)$, moreover, ensures that $ac\neq 0$. But then again the theory of binary quadratic equations ensures that the number of solutions of the equation $g_i(z^2,w^2)=0$, with $1\leqslant z,w\leqslant P_j$, counted by R_1 is once more at most $O(P^{\varepsilon})$.

Combining the conclusions of the previous four paragraphs, we find that for every fixed choice of $\mathbf{h}$, $\mathbf{m}$, $\mathbf{n}$ satisfying (3.8) for which the equation (3.9) does not hold, the number of possible choices for z and w satisfying (3.7) is at most $O(P^{2/3+\varepsilon})$. Consequently,

$$R_1\ll P^{2/3+\varepsilon}\widetilde{H}_j\widetilde{M}_j^2. \tag{3.21}$$

We therefore conclude from (3.6), (3.10), (3.17) and (3.21) that

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{1+\varepsilon}K_j(P;\boldsymbol{\phi}) + P^{2/3+\varepsilon}\widetilde{H}_j\widetilde{M}_j^2,$$

whence the lemma follows immediately.

On combining the conclusion of Lemma 3.3 with Lemma 3.1, we are able to obtain mean value estimates for 2^l-th moments of $F_j(\alpha)$ which, in many circumstances, are superior to those available hitherto. We summarise such new estimates, and recall those previously known, in the following theorem.

Theorem 3.4. *Suppose that* $1\leqslant l\leqslant k-2$ *and* $1\leqslant j\leqslant k-1-l$. *Let* $\nu=[j/2]$, $J=\left[\frac{1}{2}(k-j-l+1)\right]$ *and for* $r\geqslant 1$ *write* $\delta_r=\lambda_r^{(2Jk)}-r$. *Suppose that* δ_r *is increasing with* r, *and let* e *be* 0 *or* 1 *according to whether* j *is even or odd. Finally, define the exponent* σ *in general by taking* $\sigma=\delta_j/j$, *and when* $(k+\delta_{2(\nu+f)}-2\delta_{\nu+f})\phi_1\leqslant 1$ $(f=0,e)$, *by taking* $\sigma=(\delta_\nu+\delta_{\nu+e})/j$. *Then the following hold.*

(Ia) *Unconditionally, if* $j=1$, *or*
(Ib) *if any one of the following conditions hold,*
(α) $l=1$ *and* $k-j$ *is either odd, or* $k-j=2$ *or* 4, *or*
(β) $l=2$ *and* $3\leqslant k-j\leqslant 5$, *or*
(γ) $l=3$ *and* $k-j=4$ *or* 5, *or*
(δ) $\phi_1+\dots+\phi_j\leqslant\frac{1}{3}$,
and in addition any one of the following conditions also hold,
(i) $1\leqslant j\leqslant J+1$, *or*
(ii) $2+e\leqslant j\leqslant 2J+2-e$ *and* $(k+\delta_{j+e})\phi_1\leqslant 1$, *or*
(iii) *when* $j\geqslant 3$, *we have*

$$\sum_{i=1}^{I}\phi_i+k(\phi_{I-1}+\phi_I)\leqslant 2\quad(3\leqslant I\leqslant j),$$

then one has

$$\int_0^1|F_j(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l-1}\widetilde{H}_j^{2^l-1}.$$

(Ic) *If any one of the conditions* (α), (β), (γ), *or*
(δ') $\phi_1+\dots+\phi_j\leqslant\frac{1}{3}(1-\sigma)^{-1}$,
hold, and none of (i), (ii), (iii) *hold, then one has*

$$\int_0^1|F_j(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l-1+\sigma}\widetilde{H}_j^{2^l-1}.$$

(II) *If*
(1) *one of conditions* (i), (ii), (iii) *hold, and* $\phi_1+\dots+\phi_j\geqslant\frac{1}{3}$, *or*
(2) *none of conditions* (i), (ii), (iii) *hold, and* $\phi_1+\dots+\phi_j\geqslant\frac{1}{3}(1-\sigma)^{-1}$,
then

$$\int_0^1|F_j(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l-\frac{1}{3}+\varepsilon}\widetilde{M}_j^{2^l}\widetilde{H}_j^{2^l-1}.$$

(III) In any case, one has

$$\int_0^1 |F_j(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l}\widetilde{H}_j^{2^l-1}.$$

Proof. For the sake of convenience, write $m = l-1$. We begin by noting that $M_1 \leqslant P^{1/3}$ for $k \geqslant 3$, so that by combining the conclusions of Lemma 2.1 of Vaughan [14], Lemma 3.3 of [18], and Lemma 3.3 of the present paper, we conclude for $j=1$ that

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j \quad (0\leqslant r\leqslant m). \tag{3.22}$$

Suppose next that one of the conditions (i), (ii) or (iii) holds. Then the argument of the proofs of Theorems 3.10 and 3.11 of [18] shows that

$$K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j \quad (0\leqslant r\leqslant m). \tag{3.23}$$

When conditions (α), (β) or (γ) hold, it follows that for $0\leqslant r\leqslant m$, the integer $k-j-r$ is either odd, or else is equal either to 2 or 4. In these circumstances, Lemma 3.3 of [18] shows that

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \quad (0\leqslant r\leqslant m), \tag{3.24}$$

whence the estimate (3.22) follows from (3.23). When condition (δ) holds, on the other hand, one has $\widetilde{M}_j \ll P^{1/3}$, and so we may conclude from Lemma 3.3 of the present paper together with Lemma 3.3 of [18] that for $0\leqslant r\leqslant m$,

$$\begin{aligned} R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) &\ll P^{1+\varepsilon}K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) + P^{\frac{2}{3}+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j^2 \\ &\ll P^{1+\varepsilon}K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) + P^{1+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j. \end{aligned}$$

Here we note that the former lemma applies when $k-j-r$ is an even integer exceeding 4, and the latter when $k-j-r$ is odd, or equal to 2 or 4. Thus the estimate (3.22) again follows from (3.23).

When none of the conditions (i), (ii), (iii) hold, meanwhile, then the argument of the proofs of Theorems 3.10 and 3.11 of [18] yields the estimate

$$K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{r+\varepsilon}\widetilde{H}_j\widetilde{M}_j^{1+\sigma}. \tag{3.25}$$

Since (3.24) again holds when conditions (α), (β) or (γ) are satisfied, we deduce from (3.25) that when one of the latter conditions holds, one has

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j^{1+\sigma} \quad (0\leqslant r\leqslant m). \tag{3.26}$$

When condition (δ') holds, meanwhile, one has $\widetilde{M}_j^{1-\sigma} \ll P^{1/3}$, and in such circumstances one may conclude from Lemma 3.3 of the present paper together with Lemma 3.3 of [18] that for $0\leqslant r\leqslant m$,

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}K_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) + P^{\frac{2}{3}+\varepsilon}(P^r\widetilde{H}_j)(\widetilde{M}_j^{1+\sigma}P^{\frac{1}{3}}),$$

and hence (3.26) again follows from (3.25).

When condition (1) of the statement of Theorem 3.4 holds, one has $\widetilde{M}_j \gg P^{1/3}$, and thus from Lemma 3.3 of the present paper, Lemma 3.3 of [18], and (3.23), one obtains in a manner similar to that above,

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{\frac{2}{3}+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j^2 \quad (0\leqslant r\leqslant m). \tag{3.27}$$

When condition (2) of the statement of Theorem 3.4 holds, meanwhile, we have $\widetilde{M}_j^{1-\sigma} \gg P^{1/3}$, whence by Lemma 3.3 of the present paper, Lemma 3.3 of [18], and (3.25), one obtains the estimate (3.27) once again. Finally, we note that Lemma 3.2 of [18] provides the bound

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j^2 \quad (0\leqslant r\leqslant m). \tag{3.28}$$

On collecting together (3.22), (3.26) and (3.28), we find that in all cases one has a bound of the shape

$$R^{(1)}_{j+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}(P^r\widetilde{H}_j)\widetilde{M}_j^{1+\tau} \quad (0\leqslant r\leqslant m), \tag{3.29}$$

where $\tau = 0$ when conditions (Ia) or (Ib) hold, where $\tau = \sigma$ when (Ic) holds, where $\widetilde{M}_j^\tau = \widetilde{M}_j P^{-1/3}$ when (II) holds, and where $\tau = 1$ when (III) holds. We now apply Lemma 3.1, obtaining from (3.29) for $1\leqslant r\leqslant m$ the estimate

$$\begin{aligned} R_j^{(2^r)}(P;\boldsymbol{\phi}) \ll & P^{2^r-1}(\widetilde{H}_j\widetilde{M}_j)^{2^r}R_j^{(2^{r-1})}(P;\boldsymbol{\phi}) \\ & + P^{2^{r+1}-2r-2}(\widetilde{H}_j\widetilde{M}_j)^{2^{r+1}-2}\left(P^{r+1+\varepsilon}\widetilde{H}_j\widetilde{M}_j^{1+\tau}\right) \\ \ll & P^{2^{r+1}-r-1+\varepsilon}\widetilde{H}_j^{2^{r+1}-1}\widetilde{M}_j^{2^{r+1}-1+\tau} \\ & + P^{2^r-1}(\widetilde{H}_j\widetilde{M}_j)^{2^r}R_j^{(2^{r-1})}(P;\boldsymbol{\phi}). \end{aligned} \tag{3.30}$$

Then by inductively applying the formula (3.30), starting from the base

$$R_j^{(1)}(P;\boldsymbol{\phi}) \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j^{1+\tau}$$

supplied by (3.29), one deduces that for $0\leqslant r\leqslant m$, one has

$$R_j^{(2^r)}(P;\boldsymbol{\phi}) \ll P^{2^{r+1}-r-1+\varepsilon}\widetilde{H}_j^{2^{r+1}-1}\widetilde{M}_j^{2^{r+1}-1+\tau}.$$

The conclusion of the theorem follows from the case $r = m$ of the latter formula, on considering the underlying diophantine equations.

For the sake of completeness we add a final mean value estimate related to those of Theorem 3.4 to our arsenal.

Theorem 3.5. *Suppose that* $2\leqslant l\leqslant k-2$. *Then one has*

$$\int_0^1 |F_{k-l}(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l+\varepsilon}\widetilde{M}_{k-l}^{2^l}\widetilde{H}_{k-l}^{2^l-1}.$$

Proof. On considering the underlying diophantine equations, the argument of the proof of case (III) of Theorem 3.4 leading to (3.29) yields

$$R^{(1)}_{k-l+r}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{1+\varepsilon}(P^r\widetilde{H}_{k-l})\widetilde{M}_{k-l}^2 \quad (0\leqslant r\leqslant l-2). \tag{3.31}$$

Moreover, it follows from (3.1) and the definition of $R^{(1)}_{k-1}(P;\boldsymbol{\phi},\mathbf{0})$ that

$$R^{(1)}_{k-1}(P;\boldsymbol{\phi},\mathbf{0})\leqslant\widetilde{M}_{k-l}^2R^*, \tag{3.32}$$

where R^* denotes the number of integral solutions of the equation

$$z_1\dots z_lh_1\dots h_{k-l} = w_1\dots w_lg_1\dots g_{k-l}, \tag{3.33}$$

with $1\leqslant z_i,w_i\leqslant P_{k-l}$ $(1\leqslant i\leqslant l)$ and $1\leqslant h_n,g_n\leqslant 2^kH_n$ $(1\leqslant n\leqslant k-l)$. Let $\mathbf{z},\mathbf{w},\mathbf{g},\mathbf{h}$ be a solution of (3.33) counted by R^*. Standard estimates for the divisor function reveal that for each fixed choice of $\mathbf{z},\mathbf{h}$, one has $O(P^\varepsilon)$ possible choices for $\mathbf{w},\mathbf{g}$, whence $R^*=O(P^{l+\varepsilon}\widetilde{H}_{k-l})$. By (3.32), we therefore have

$$R^{(1)}_{k-1}(P;\boldsymbol{\phi},\mathbf{0}) \ll P^{l+\varepsilon}\widetilde{M}_{k-l}^2\widetilde{H}_{k-l},$$

so that (3.31) holds also when $r=l-1$. The lemma now follows by applying Lemma 3.1 inductively in the same manner as in the proof of Theorem 3.4.

4. Iterative schemes based on mean value estimates

In our mean value based treatments we adopt two approaches, according to the situation. We consider below the consequences of estimates of the form

$$\int_0^1 |F_j(\alpha)|^{2^l}\,d\alpha \ll P^{2^l-l-\chi_{j,l}+\varepsilon}\widetilde{M}_j^{2^l-1+\tau_{j,l}}\widetilde{H}_j^{2^l-1}, \tag{4.1}$$

for a suitable $\tau_{j,l}\geqslant 0$, and $\chi_{j,l}=0$ or $\frac{1}{3}$. We suppose in what follows that λ_r $(r\in\mathbb{N})$ are known permissible exponents, and we seek a new permissible exponent λ_s'.

(i) Process $A_j^{s,l}$. When $s\geqslant j$ we may adopt the scheme

$$\begin{array}{lllllll}
F_0^2f_0^{2s-2} & & & & & & \\
\downarrow & & & & & & \\
F_1f_1^{2s-2} & \to F_2f_2^{2s-4} & \to F_3f_3^{2s-6} & \to\cdots\to & F_jf_j^{2s-2j} & \Rightarrow & (F_j^{2^l})^{2^{-l}}(f_j^{2t-2})^{a_s}(f_j^{2t})^{b_s} \\
 & \quad\downarrow & \quad\downarrow & & \quad\downarrow & & \\
 & \quad f_1^{2s} & \quad f_2^{2s-2} & & f_{j-1}^{2s-2j+4} & &
\end{array}$$

where t, a_s and b_s are defined by means of

$$t=[(1-2^{-l})^{-1}(s-j)+1],\quad \theta=t-(1-2^{-l})^{-1}(s-j),$$
$$a_s=(1-2^{-l})\theta,\quad b_s=(1-2^{-l})(1-\theta).$$

Following the argument of §2 of [18] (see also §11), and recalling the definitions of the parameters from §2, it follows from (4.1) that λ_s' and $\boldsymbol{\phi}$ are determined by the equations

$$P\widetilde{H}_{j-1}\widetilde{M}_jQ_j^{\lambda_{s-j}}\approx P^{1-(l+\chi_{j,l})2^{-l}}\widetilde{H}_j^{1-2^{-l}}\widetilde{M}_j^{1-2^{-l}(1-\tau_{j,l})}Q_j^{a_s\lambda_{t-1}+b_s\lambda_t},$$
$$P\widetilde{H}_{i-1}\widetilde{M}_iQ_i^{\lambda_{s-i}}\approx\Big(P(\widetilde{H}_i\widetilde{M}_i)^2M_{i+1}^{2(s-i-1)}Q_i^{\lambda_{s-i+1}}Q_{i+1}^{\lambda_{s-i-1}}\Big)^{1/2}\quad(2\leqslant i\leqslant j-1),\tag{4.2}$$

$$PM_1Q_1^{\lambda_{s-1}}\approx\Big(P(H_1M_1)^2M_2^{2s-4}Q_1^{\lambda_s}Q_2^{\lambda_{s-2}}\Big)^{1/2},\tag{4.3}$$

$$P^{\lambda_s'}\approx PM_1^{2s-2}Q_1^{\lambda_{s-1}}.\tag{4.4}$$

Here and throughout, we use the symbol $\approx$ to denote that factors involving R and P^ε to fixed powers are to be ignored.

Write

$$\delta=(2^l-1)(\theta\lambda_{t-1}+(1-\theta)\lambda_t)-2^l\lambda_{s-j},$$
$$\mathcal{E}_i=\lambda_{s-i+1}-2\lambda_{s-i}+\lambda_{s-i-1}\quad(1\leqslant i<j),\tag{4.5}$$
$$\kappa_i=2(s-i)-\lambda_{s-i}\quad(2\leqslant i\leqslant j).\tag{4.6}$$

Now define α_i, β_i, γ_i for $1\leqslant i\leqslant j$ by

$$\alpha_j=((2^l-1)k+\delta+1-\tau_{j,l})^{-1},$$
$$\beta_j=-k+\delta+1-\tau_{j,l},$$
$$\gamma_j=2^l-l+\delta-j-\chi_{j,l},$$

and for $i=j-1,\dots,1$, successively by

$$\begin{aligned}\gamma_i&=1+\mathcal{E}_i+\kappa_{i+1}\alpha_{i+1}\gamma_{i+1},\\ \beta_i&=\mathcal{E}_i+\kappa_{i+1}\alpha_{i+1}\beta_{i+1},\\ \alpha_i&=(2k+\beta_i)^{-1}.\end{aligned}\tag{4.7}$$

Then on writing explicitly the equations relating the ϕ_i described above, and solving the resulting system of linear equations, one verifies with little difficulty that $\boldsymbol{\phi}$ and λ_s' satisfy

$$\phi_i=\alpha_i\left(\gamma_i-\beta_i(\phi_1+\dots+\phi_{i-1})\right)\quad(2\leqslant i\leqslant j),\tag{4.8}$$
$$\phi_1=\alpha_1\gamma_1,\tag{4.9}$$

and

$$\lambda_s'=\lambda_{s-1}(1-\phi_1)+1+(2s-2)\phi_1.\tag{4.10}$$

In this manner we may calculate a new permissible exponent λ_s', and in concert with other available iterative schemes we repeatedly derive new sequences (λ_r) of permissible exponents, ultimately attaining an approximation to converged values (see the discussion of §2 of [18] for a detailed overview of such matters).

(ii) Process $B_{j,t}^{s,l}$. When $s\geqslant j$ and

$$(1-2^{-l})^{-1}(s-j)\leqslant t\leqslant(1-2^{1-l})^{-1}(s-j),$$

we may instead adopt the scheme

$$\begin{array}{llllll} F_0^2 f_0^{2s-2} & & & & & \\ \downarrow & & & & & \\ F_1 f_1^{2s-2} & \to F_2 f_2^{2s-4} & \to F_3 f_3^{2s-6} & \to \cdots \to & F_j f_j^{2s-2j} & \Rightarrow (F_j^{2^{l-1}})^{a_s}(F_j^{2^l})^{b_s}(f_j^{2t})^{\frac{s-j}{t}} \\ & \downarrow & \downarrow & & \downarrow & \\ & f_1^{2s} & f_2^{2s-2} & & f_{j-1}^{2s-2j+4} & \end{array}$$

where a_s and b_s are defined by

$$a_s=2-2^{1-l}-2(s-j)t^{-1},\quad b_s=(s-j)t^{-1}-1+2^{1-l}.$$

Following the argument of §2 of [18] (see also §11), it follows from (4.1) that λ_s' and $\boldsymbol{\phi}$ are determined by the equations

$$P\widetilde{H}_{j-1}\widetilde{M}_jQ_j^{\lambda_{s-j}}\approx P^{(2^{l-1}-l+1-\chi_{j,l-1})a_s+(2^l-l-\chi_{j,l})b_s}(\widetilde{H}_j\widetilde{M}_j)^{(2^{l-1}-1)a_s+(2^l-1)b_s}$$
$$\times\widetilde{M}_j^{a_s\tau_{j,l-1}+b_s\tau_{j,l}}Q_j^{\lambda_t(s-j)/t},$$

and the equations (4.2)-(4.4).

We now write

$$\delta=2(s-j)\lambda_t-2t\lambda_{s-j},$$

and define $\mathcal{E}_i$ and κ_i as in (4.5) and (4.6). Also, we define in this case α_i, β_i, γ_i for $1\leqslant i\leqslant j$ by means of

$$\begin{aligned}\alpha_j&=(2(s-j)(k-1)+2t+\delta+\tilde{\tau})^{-1},\\ \beta_j&=2(k-1)(s-j-t)+\delta+\tilde{\tau},\\ \gamma_j&=2(j+l-2)(s-j-t)+t(2-2^{2-l})+\delta-\tilde{\chi},\end{aligned}$$

in which

$$\tilde{\tau}=(4(s-j)-(4-2^{2-l})t)\tau_{j,l-1}+((2-2^{2-l})t-2(s-j))\tau_{j,l},$$

$$\tilde{\chi}=((4-2^{2-l})t-4(s-j))\chi_{j,l-1}+(2(s-j)-t(2-2^{2-l}))\chi_{j,l},$$

and for $i=j-1,\dots,1$, successively by (4.7). Then we find once again that $\boldsymbol{\phi}$ and λ_s' satisfy (4.8)-(4.10), and once more we are able to establish new permissible exponents by iterating this and allied procedures.

Notice that both processes $A_j^{s,l}$ and $B_{j,t}^{s,l}$ apply in particular when $j=1$, in which case they may or may not duplicate the methods of Vaughan [13, 14].

5. Iterative schemes based on the Hardy-Littlewood method

We next investigate estimates arising from iterative schemes of the shape

$$\begin{array}{lllllll}
F_0^2 f_0^{2s-2} & & & & & & \\
\quad\downarrow & & & & & & \\
F_1 f_1^{2s-2} & \to F_2 f_2^{2s-4} & \to F_3 f_3^{2s-6} & \to \cdots \to & F_j f_j^{2s-2j} & \Rightarrow (F_j)(f_j^{2s-2j}) & \qquad (M_1) \\
 & \quad\downarrow & \quad\downarrow & & \quad\downarrow & & \\
 & \quad f_1^{2s} & \quad f_2^{2s-2} & & f_{j-1}^{2s-2j+4} & &
\end{array}$$

and

$$\begin{array}{lllll}
F_0^2 f_0^{+\,2s-2} \to F_1 f_1^{+\,2s-2} & \to F_2 f_2^{+\,2s-4} & \to F_3 f_3^{+\,2s-6} & \to \dots & \\
 & \quad\downarrow & \quad\downarrow & & \\
 & \quad f_1^{2s} & \quad f_2^{2s-2} & & \\
\cdots \to F_{j-1} f_{j-1}^{+\,2s-2j+2} & \to F_j g_j^2 f_j^{2s-2j-2} & \Rightarrow (F_j)(g_j^2 f_j^{2s-2j-2}) & & \\
\qquad\quad\downarrow & \qquad\downarrow & & & \qquad (M_2) \\
\qquad f_{j-2}^{2s-2j+6} & \quad f_{j-1}^{2s-2j+4} & & &
\end{array}$$

A perusal of the arguments of [18] should convince the reader that the derivation of bounds close to optimal via the Hardy-Littlewood method in such schemes is a matter of considerable complexity. We therefore strive for simplicity, sacrificing a little on performance.

In the first iterative scheme above, we estimate the mean value occurring in the final step of the iterative procedure by means of Lemma 13.1 of [18], which we record in a slightly more general form. We first require some notation.

Definition 5.1. *Suppose that* $k\geqslant 4$ *and* $1\leqslant j\leqslant k-3$.

(i) Let $\mathfrak{M}_j$ *denote the union of the intervals*

$$\mathfrak{M}_j(q,a)=\{\alpha\in[0,1)\ :\ |q\alpha-a|\leqslant PQ_j^{-k}R^{k(j-k)}\},$$

with $0\leqslant a\leqslant q\leqslant P$ *and* $(a,q)=1$. *Also, let* $\mathfrak{m}_j=[0,1)\setminus\mathfrak{M}_j$.

(ii) Define ϖ_j *to be* 0 *when* $j=k-3$, *and to be* 1 *when* $1\leqslant j\leqslant k-4$. *Also, write* $w_j=2^{1+j-k}$.

(iii) Let $\mathfrak{N}_j$ *denote the union of the intervals*

$$\mathfrak{N}_j(q,a)=\{\alpha\in[0,1)\ :\ |q\alpha-a|\leqslant(PM_1^{\varpi_j})^{w_j(k-j)}Q_j^{-k}\},$$

with $0\leqslant a\leqslant q\leqslant(PM_1^{\varpi_j})^{w_j(k-j)}$ *and* $(a,q)=1$.

We note that the $\mathfrak{M}_j(q,a)$ comprising $\mathfrak{M}_j$ are disjoint, and likewise also the $\mathfrak{N}_j(q,a)$ comprising $\mathfrak{N}_j$.

Lemma 5.2. *Suppose that* $k\geqslant 4$ *and* $1\leqslant j\leqslant k-3$. *Let* u *be a positive integer, and define*

$$t=\left[\left(\frac{k-j+1}{k-j}\right)u+1\right],\quad \theta=t-\left(\frac{k-j+1}{k-j}\right)u.$$

Suppose that Δ_{t-1} and Δ_t are admissible exponents, and write

$$\mu_u = \frac{k-j}{k-j+1}(\theta\Delta_{t-1} + (1-\theta)\Delta_t).$$

Then

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_jQ_j^{2u-k}\left((PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_u} + Q_j^{\mu_u}\right).$$

Proof. When $1\leqslant j\leqslant k-4$, the stated conclusion is provided by Lemma 13.1 of [18]. When $j = k-3$, meanwhile, the conclusion follows from the argument of the proof of the latter lemma, noting that by a Weyl differencing argument paralleling those of Lemmata 6.1 and 12.1 of [18], it follows from Lemma 4.1 of [18] together with a trivial estimate that

$$\sup_{\alpha\in\mathfrak{m}_{k-3}} |F_{k-3}(\alpha)| \ll P^{1-w_{k-3}+\varepsilon}\widetilde{H}_{k-3}\widetilde{M}_{k-3}.$$

We next consider the second of the iterative schemes above, but in order to make further progress we require some additional notation. When $k\geqslant 2$, we write

$$S_k(q,a) = \sum_{r=1}^{q} e(ar^k/q),$$

and define also the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-\frac{1}{2}}, & \text{when } u\geqslant 0 \text{ and } v=1,\\ p^{-u-1}, & \text{when } u\geqslant 0 \text{ and } 2\leqslant v\leqslant k.\end{cases}$$

Note that there is some possibility of confusion between the function $w_k(q)$ and the exponent w_j, but that a perusal of the context should easily dispel any ambiguity. Then according to Lemma 3 of Vaughan [12], whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, one has

$$q^{-1/2}\leqslant w_k(q) \ll q^{-1/k}, \tag{5.1}$$

and

$$q^{-1}S_k(q,a) \ll w_k(q). \tag{5.2}$$

We require the estimate contained in the following lemma.

Lemma 5.3. *Suppose that $k\geqslant 4$ and $1\leqslant j\leqslant k-3$. Then whenever $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$, one has*

$$g_j(\alpha) \ll Q_j^{1+\varepsilon}w_k(q)(1+Q_j^k|\alpha - a/q|)^{-1} + P^{\varepsilon}(PM_1^{\varpi_j})^{\frac{1}{2}w_j(k-j)}.$$

In particular, whenever $Q_j \geqslant (PM_1^{\varpi_j})^{w_j(k-j)}$*, one has*

$$g_j(\alpha) \ll Q_j^{1+\varepsilon} w_k(q)(1+Q_j^k|\alpha - a/q|)^{-1}.$$

Proof. On making use of the refinements embodied in Theorem 4.1 of Vaughan [15], we deduce that whenever $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$, one has

$$g_j(\alpha) - q^{-1}S_k(q,a)v_j^+(\beta) \ll q^{\frac{1}{2}+\varepsilon}(1+Q_j^k|\alpha - a/q|)^{1/2},$$

where

$$v_j^+(\beta) = \int_{\frac{1}{2}Q_jR^{-j}}^{Q_j} e(\beta\gamma^k)d\gamma.$$

By partial integration, one readily deduces that

$$v_j^+(\beta) \ll \min\{Q_j, (Q_jR^{-j})^{1-k}|\beta|\} \ll \frac{Q_jR^{j(k-1)}}{1+Q_j^k|\beta|}.$$

On recalling (5.2), therefore, we deduce that for $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$, one has

$$g_j(\alpha) \ll \frac{w_k(q)Q_j^{1+\varepsilon}}{1+Q_j^k|\alpha - a/q|} + P^\varepsilon (PM_1^{\varpi_j})^{\frac{1}{2}w_j(k-j)}. \tag{5.3}$$

This establishes the first conclusion of the lemma. When $Q_j \geqslant (PM_1^{\varpi_j})^{w_j(k-j)}$, it follows from (2.1) and (5.1) that for $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$, the first term on the right hand side of (5.3) majorises the second, up to a factor of P^ε. The second conclusion of the lemma is now immediate.

We must also estimate $F_j(\alpha)$ for $\alpha \in \mathfrak{M}_j$ in order to prosecute the estimation required for the use of the second iterative scheme. In this context, we write

$$\tau_j(q,a,\mathbf{h},\mathbf{m}) = \Big|\sum_{r=1}^{q} e\left(\frac{a}{q}\Psi_j(r,\mathbf{h},\mathbf{m})\right)\Big|,$$

and then define $F_j^*(\alpha)$ to be zero whenever $\alpha \in \mathfrak{m}_j$, and by

$$F_j^*(\alpha) = \sum_{\mathbf{m}}\sum_{\mathbf{h}} \frac{Pq^{-1}\tau_j(q,a,\mathbf{h},\mathbf{m})}{(1+|\alpha - a/q|h_1\dots h_jP^{k-j})^{1/(k-j)}},$$

when $\alpha \in \mathfrak{M}_j(q,a) \subseteq \mathfrak{M}_j$. Here, the summation is over $\mathbf{m}$ and $\mathbf{h}$ satisfying (2.1). Finally, we define $g_j^*(\alpha)$ to be zero for $\alpha \in \mathfrak{n}_j$, and by

$$g_j^*(\alpha) = Q_j^{1+\varepsilon}w_k(q)(1+Q_j^k|\alpha - a/q|)^{-1},$$

when $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$. We observe that this definition of $g_j^*(\alpha)$ differs from that provided in §2 of Vaughan and Wooley [16], but not in a manner damaging to our subsequent argument.

We now describe an auxiliary lemma which may be of interest beyond this work. Our treatment here is motivated by the proof of Lemma 3.1 of Brüdern and Wooley [4].

Lemma 5.4. *Suppose that* $k \geqslant 4$. *Let* Q *be a real number with* $1 \leqslant Q \leqslant P$. *Let* $\mathfrak{M}$ *denote the union of the intervals*

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) \ : \ |q\alpha - a| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ *and* $(a,q) = 1$. *Let* δ *be a real number with* $\delta > 1$, *and define the function* $\Upsilon(\alpha)$ *for* $\alpha \in \mathfrak{M}$ *by taking*

$$\Upsilon(\alpha) = w_k(q)^2 (1 + P^k|\alpha - a/q|)^{-\delta}$$

when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$. *Also, write* $t = \left[\frac{1}{2}k\right]$. *Then for any subset* $\mathcal{A}$ *of* $[1,P] \cap \mathbb{Z}$, *one has for each* $\varepsilon > 0$ *the estimate*

$$\int_{\mathfrak{M}} \Upsilon(\alpha) \Big| \sum_{x \in \mathcal{A}} e(\alpha x^k) \Big|^{2t} d\alpha \ll Q^{\varepsilon} P^{2t-k}.$$

Proof. We begin by observing that

$$\int_{\mathfrak{M}} \Upsilon(\alpha) \Big| \sum_{x \in \mathcal{A}} e(\alpha x^k) \Big|^{2t} d\alpha \leqslant \sum_{1 \leqslant q \leqslant Q} w_k(q)^2 \int_{-Q/P^k}^{Q/P^k} (1 + P^k|\beta|)^{-\delta} \sum_{a=1}^{q} \Big| \sum_{x \in \mathcal{A}} e(x^k(\beta + a/q)) \Big|^{2t} d\beta. \tag{5.4}$$

By orthogonality,

$$\sum_{a=1}^{q} \Big| \sum_{x \in \mathcal{A}} e(x^k(\beta + a/q)) \Big|^{2t} = q \sum_{\substack{\mathbf{x} \in \mathcal{A}^{2t} \\ q | \psi(\mathbf{x})}} e(\beta \psi(\mathbf{x})), \tag{5.5}$$

where we write

$$\psi(\mathbf{x}) = \sum_{i=1}^{t} (x_{2i-1}^k - x_{2i}^k). \tag{5.6}$$

But plainly,

$$\sum_{\substack{\mathbf{x} \in \mathcal{A}^{2t} \\ q | \psi(\mathbf{x})}} e(\beta \psi(\mathbf{x})) \leqslant \sum_{\substack{1 \leqslant x_1, \ldots, x_{2t} \leqslant P \\ q | \psi(\mathbf{x})}} 1 \leqslant (Pq^{-1} + 1)^{2t} \rho(q), \tag{5.7}$$

where $\rho(q)$ denotes the number of solutions of the congruence

$$\sum_{i=1}^{t} (x_{2i-1}^k - x_{2i}^k) \equiv 0 \pmod{q},$$

with $1\leqslant x_i\leqslant q$ $(1\leqslant i\leqslant 2t)$. By orthogonality, moreover, it follows from (5.6) that

$$q\rho(q)=\sum_{a=1}^q |S_k(q,a)|^{2t}=\sum_{a=1}^q (q,a)^{2t}\Big| S_k\Big(\frac{q}{(q,a)},\frac{a}{(q,a)}\Big)\Big|^{2t},$$

whence by (5.2),

$$q\rho(q)\ll q^{2t}\sum_{a=1}^q w_k(q/(q,a))^{2t}=q^{2t}\sum_{r|q} rw_k(r)^{2t}.$$

Consequently, on inserting this estimate into (5.7) and substituting into (5.4) and (5.5), we deduce that

$$\int_{\mathfrak{M}} \Upsilon(\alpha)\Big|\sum_{x\in\mathcal{A}} e(\alpha x^k)\Big|^{2t}d\alpha \ll P^{2t-k}\sum_{1\leqslant q\leqslant Q} w_k(q)^2\sigma(q), \tag{5.8}$$

where

$$\sigma(q)=\sum_{r|q} rw_k(r)^{2t}. \tag{5.9}$$

The function $w_k(r)$ is multiplicative with respect to r, and thus $\sigma(q)$ is likewise a multiplicative function of q. Further, it follows from (5.9) that for each prime p and natural number h, one has

$$\sigma(p^h)=\sum_{l=0}^h p^l w_k(p^l)^{2t},$$

whence by the definition of $w_k(q)$,

$$\sigma(p^h)=1+\sum_{uk+1\leqslant h} p^{uk+1}(kp^{-u-\frac12})^{2t}+\sum_{v=2}^k\sum_{uk+v\leqslant h} p^{u(k-2t)+v-2t}.$$

Thus, on recalling that $t=[k/2]$, we deduce that

$$\sigma(p)\leqslant 1+k^{2t}p^{-1},$$

and for $h\geqslant 2$ we obtain

$$\begin{aligned}\sigma(p^h)&\ll_k p^{\frac{h-1}{k}(k-2t)+1-t}+\sum_{v=2}^k p^{\frac{h-v}{k}(k-2t)+v-2t}\\ &\ll_k p^{\frac{h-1}{k}(k-2t)+1-t}+p^{\frac{h}{k}(k-2t)}\ll p^{h/k}.\end{aligned}$$

We therefore arrive at the estimates

$$\begin{aligned} w_k(p)^2\sigma(p) &\ll_k p^{-1}, \\ w_k(p^{uk+1})^2\sigma(p^{uk+1}) &\ll_k p^{-u-1+\frac{1}{k}} \quad (u\geqslant 1), \\ w_k(p^{uk+v})^2\sigma(p^{uk+v}) &\ll_k p^{-u-1} \quad (u\geqslant 0 \text{ and } 2\leqslant v\leqslant k). \end{aligned}$$

The multiplicative properties of $\sigma(q)$ and $w_k(q)$ consequently assure us that for a suitable constant A depending at most on k,

$$\begin{aligned} \sum_{1\leqslant q\leqslant Q} w_k(q)^2\sigma(q) &\leqslant \prod_{p\leqslant Q}\Big(1+\sum_{h=1}^{\infty} w_k(p^h)^2\sigma(p^h)\Big) \\ &\leqslant \prod_{p\leqslant Q}(1+Ap^{-1}) \ll Q^{\varepsilon}. \end{aligned}$$

The conclusion of the lemma now follows immediately from (5.8).

We record an immediate corollary of Lemma 5.4 in the form of the following lemma.

Lemma 5.5. *Suppose that* $k\geqslant 4$, $1\leqslant j\leqslant k-3$ *and* $u\geqslant[\frac{1}{2}k]$. *Suppose also that*

$$Q_j\geqslant (PM^{\varpi_j})^{w_j(k-j)}.$$

Then

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{M}_j\widetilde{H}_jQ_j^{2u+2-k}\mathcal{M},$$

where

$$\mathcal{M}=(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}+1.$$

Proof. Following the argument of the proof of Lemma 3.1 of [16] (see, in particular, equations (3.1), (3.7) and (3.8) of that paper), we obtain

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_jQ_j^{2u+2-k}(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}+I, \quad (5.10)$$

where

$$I=\int_{\mathfrak{N}_j} F_j^*(\alpha)|g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha. \quad (5.11)$$

We note here that our choice of ϖ_j when $j=k-3$ ensures that the above conclusion remains valid also when $j=k-3$. Since by hypothesis we have

$$(PM_1^{\varpi_j})^{w_j(k-j)}\leqslant Q_j,$$

it follows from Lemma 5.3 that for $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$, one has

$$g_j(\alpha) \ll Q_j^{1+\varepsilon} w_k(q)(1+Q_j^k|\alpha - a/q|)^{-1}.$$

Then on making use of a trivial estimate for $F_j^*(\alpha)$, we deduce from (5.11) that

$$I \ll P\widetilde{M}_j\widetilde{H}_jQ_j^{2+\varepsilon}\int_{\mathfrak{N}_j}\Upsilon_j(\alpha)|f_j(\alpha)|^{2u}d\alpha, \tag{5.12}$$

where $\Upsilon_j(\alpha)$ is the function defined for $\alpha \in \mathfrak{N}_j$ by taking

$$\Upsilon_j(\alpha) = w_k(q)^2(1+Q_j^k|\alpha - a/q|)^{-2},$$

when $\alpha \in \mathfrak{N}_j(q,a) \subseteq \mathfrak{N}_j$. But the hypotheses of the statement of Lemma 5.4 are satisfied for $\Upsilon_j(\alpha)$ on $\mathfrak{N}_j$, whence for $u \geqslant [\frac{1}{2}k]$ we find that

$$\int_{\mathfrak{N}_j}\Upsilon_j(\alpha)|f_j(\alpha)|^{2u}d\alpha \ll Q_j^{2u-k+\varepsilon}. \tag{5.13}$$

Thus we conclude from (5.12) and (5.13) that

$$I \ll P^{1+\varepsilon}\widetilde{M}_j\widetilde{H}_jQ_j^{2u+2-k},$$

and hence the desired conclusion is immediate from (5.10).

We supplement Lemma 5.5 with a variant of Lemmata 3.1 and 3.2 of [16] which is of interest when

$$(PM_1^{\varpi_j})^{\frac{1}{2}w_j(k-j)} < Q_j < (PM_1^{\varpi_j})^{w_j(k-j)}. \tag{5.14}$$

Lemma 5.6. *Suppose that $k \geqslant 4$ and $1 \leqslant j \leqslant k-3$, and suppose also that the condition (5.14) holds. Let u be a positive integer, and define t and θ as in the statement of Lemma 5.2. Suppose further that Δ_{t-1} and Δ_t are admissible exponents, and define μ_u also as in the statement of Lemma 5.2. Define next*

$$\gamma = 1 - \frac{2}{k} - \frac{1}{k-j+1}, \quad v = u - \frac{4}{k}, \quad w = [\gamma^{-1}v+1],$$

and

$$\theta' = w - \gamma^{-1}v.$$

Suppose that Δ_{w-1} and Δ_w are admissible exponents, and write

$$\rho_u = \gamma(\theta'\Delta_{w-1} + (1-\theta')\Delta_w).$$

Then one has

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_jQ_j^{2u+2-k}\mathcal{M},$$

where

$$\mathcal{M}=(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}+(PM_1^{\varpi_j})^{(k-j)w_j}Q_j^{\mu_u-2}+Q_j^{\rho_u}.$$

Proof. Following the argument of the proof of Lemma 3.1 of [16] (see, in particular, equations (3.1), (3.7), (3.8), (3.9) and (3.13)), we find that

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j Q_j^{2u+2-k}\mathcal{M}'+J_1, \tag{5.15}$$

where

$$\mathcal{M}'=(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}+(PM_1^{\varpi_j})^{(k-j)w_j}Q_j^{\mu_u-2} \tag{5.16}$$

and

$$J_1=\int_{\mathfrak{N}_j} F_j^*(\alpha)g_j^*(\alpha)^2|f_j(\alpha)|^{2u}d\alpha. \tag{5.17}$$

Here we note that our definition of $g_j^*(\alpha)$ differs from that of [16], the substitution of the present definition being permitted through the use of Lemma 5.3.

We estimate the mean value J_1 via Hölder's inequality, deducing from (5.17) that

$$J_1 \leqslant J_2^{1/(k-j+1)}J_3^{2/k}\left(J_4^{\theta'}J_5^{1-\theta'}\right)^{\gamma}, \tag{5.18}$$

where

$$J_2=\int_{\mathfrak{N}_j}F_j^*(\alpha)^{k-j+1}d\alpha,\quad J_3=\int_{\mathfrak{N}_j}g_j^*(\alpha)^k|f_j(\alpha)|^4 d\alpha,$$

$$J_4=\int_0^1 |f_j(\alpha)|^{2w-2}d\alpha,\quad J_5=\int_0^1|f_j(\alpha)|^{2w}d\alpha.$$

But since Δ_{w-1} and Δ_w are admissible exponents, one has

$$J_4 \ll Q_j^{2w-2-k+\Delta_{w-1}+\varepsilon}\quad \text{and}\quad J_5\ll Q_j^{2w-k+\Delta_w+\varepsilon}. \tag{5.19}$$

Also, it is a consequence of Lemma 4.10 of [18] that

$$J_2 \ll P^{\varepsilon}(P\widetilde{H}_j\widetilde{M}_j)^{k-j+1}Q_j^{-k}. \tag{5.20}$$

In order to estimate J_3 we first note that by (5.1), whenever $\alpha\in\mathfrak{N}_j(q,a)\subseteq\mathfrak{N}_j$, one has

$$\begin{aligned} g_j^*(\alpha)^k &= w_k(q)^kQ_j^{k+\varepsilon}(1+Q_j^k|\alpha-a/q|)^{-k}\\ &\ll Q_j^{k+\varepsilon}(q+Q_j^k|q\alpha-a|)^{-1}.\end{aligned}$$

When h is an integer, write ψ_h for the number of solutions of the equation

$$x_1^k+x_2^k-x_3^k-x_4^k=h,$$

with $x_i \in \mathcal{A}(Q_j, R)$ $(1 \leqslant i \leqslant 4)$. Then plainly,

$$|f_j(\alpha)|^4 = \sum_{|h| \leqslant 2Q_j^k} \psi_h e(\alpha h),$$

and thus it is a consequence of Lemma 2 of Brüdern [3] that

$$\int_{\mathfrak{N}_j} g_j^*(\alpha)^k |f_j(\alpha)|^4 d\alpha \ll Q_j^\varepsilon \left((PM_1^{\varpi_j})^{w_j(k-j)} \psi_0 + \sum_{h \neq 0} |\psi_h| \right). \tag{5.21}$$

But by Hua's Lemma (see, for example, Lemma 2.5 of Vaughan [15]), together with an elementary counting argument,

$$\psi_0 \ll Q_j^{2+\varepsilon} \quad \text{and} \quad \sum_{h \in \mathbb{Z}} |\psi_h| \ll Q_j^4.$$

Then on recalling that our hypotheses imply that

$$Q_j > (PM_1^{\varpi_j})^{\frac{1}{2} w_j(k-j)},$$

we conclude from (5.21) that

$$J_3 = \int_{\mathfrak{N}_j} g_j^*(\alpha)^k |f_j(\alpha)|^4 d\alpha \ll Q_j^{4+\varepsilon}. \tag{5.22}$$

On combining (5.18)-(5.20) and (5.22), we arrive at the estimate

$$J_1 \ll P^{1+\varepsilon} \widetilde{H}_j \widetilde{M}_j Q_j^{2u+2-k+\rho_u},$$

and so the conclusion of the lemma is immediate from (5.15) and (5.16).

Our next task is to assemble the estimates described above into a tool sufficiently simple to apply that it is viable to employ computationally. Our aim is to establish either that

$$\int_0^1 |F_j(\alpha) f_j(\alpha)^{2u+2}| d\alpha \ll P^{1+\varepsilon} \widetilde{H}_j \widetilde{M}_j Q_j^{2u+2-k} \left((PM_1^{\varpi_j})^{-w_j} Q_j^{\Delta_{u+1}} + 1 \right), \tag{5.23}$$

or else that

$$\int_0^1 |F_j(\alpha) g_j(\alpha)^2 f_j(\alpha)^{2u}| d\alpha \ll P^{1+\varepsilon} \widetilde{H}_j \widetilde{M}_j Q_j^{2u+2-k} \left((PM_1^{\varpi_j})^{-w_j} Q_j^{\Delta_{u+1}} + 1 \right). \tag{5.24}$$

Thus we seek to show that the mean values on the left hand sides of (5.23) and (5.24) are bounded above by the estimate for the minor arc contribution stemming from our methods, together with the "expected" major arc contribution. In order to ease our discussion, we list a number of conditions concerning the quantities μ_u, ρ_u and Δ_u defined in the statements of Lemmata 5.2, 5.5 and 5.6.

$$w_j(1+\varpi_j\phi_j)\leqslant(\Delta_{u+1}-\mu_{u+1})(1-\phi_1-\cdots-\phi_j), \tag{A_1}$$

$$\mu_{u+1}\leqslant 0, \tag{A_2}$$

$$u\geqslant[\tfrac{1}{2}k] \quad \text{and} \quad w_j(k-j)(1+\varpi_j\phi_1)\leqslant 1-\phi_1-\cdots-\phi_j, \tag{B}$$

$$\tfrac{1}{2}w_j(k-j)(1+\varpi_j\phi_1)\leqslant 1-\phi_1-\cdots-\phi_j\leqslant w_j(k-j)(1+\varpi_j\phi_1), \tag{C_1}$$

$$w_j(1+\varpi_j\phi_1)\leqslant(\Delta_{u+1}-\rho_u)(1-\phi_1-\cdots-\phi_j), \tag{C_2}$$

$$w_j(k-j+1)(1+\varpi_j\phi_j)\leqslant(2+\Delta_{u+1}-\mu_u)(1-\phi_1-\cdots-\phi_j), \tag{C_3}$$

$$w_j(k-j)(1+\varpi_j\phi_1)\leqslant(2-\mu_u)(1-\phi_1-\cdots-\phi_j), \tag{C_4}$$

$$\rho_u\leqslant 0, \tag{C_5}$$

$$\Delta_{u+1}(1-\phi_1-\cdots-\phi_j)>w_j(1+\varpi_j\phi_1), \tag{D_1}$$

$$\Delta_{u+1}(1-\phi_1-\cdots-\phi_j)\leqslant w_j(1+\varpi_j\phi_1). \tag{D_2}$$

We now summarise the conclusions of Lemmata 5.2, 5.5 and 5.6.

Lemma 5.7. *Let $k\geqslant 4$ and $1\leqslant j\leqslant k-3$.*

(I) *Suppose that condition (D_1) holds, and further that one of the conditions (B), or each of (C_1), (C_2), (C_3) holds. Then one has*

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j Q_j^{2u+2-k}(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}.$$

(I') *Suppose that the condition (A_1) holds. Then one has*

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2u+2}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j Q_j^{2u+2-k}(PM_1^{\varpi_j})^{-w_j}Q_j^{\Delta_{u+1}}.$$

(II) *Suppose that condition (D_2) holds, and further that one of the conditions (B), or each of (C_1), (C_4) and (C_5) holds. Then one has*

$$\int_0^1 |F_j(\alpha)g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j Q_j^{2u+2-k}.$$

(II') *Suppose that condition (D_2) holds, and further that condition (A_2) holds. Then one has*

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2u+2}|d\alpha \ll P^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_j Q_j^{2u+2-k}.$$

Proof. The assertions of each case of the lemma are immediate, save that in part (I') we have made the observation that the validity of condition (A_1) automatically implies that of (D_1).

Since for each i one has $0\leqslant\phi_i\leqslant 1/k$, one deduces readily that the condition (B) is satisfied automatically whenever

$$u\geqslant[\tfrac{1}{2}k] \quad \text{and} \quad 2^{1+j-k}(k+\varpi_j)\leqslant 1.$$

In particular, therefore, the condition (B) is satisfied when $u\geqslant[\frac{1}{2}k]$ and

$$j\leqslant k-4 \quad \text{and} \quad k\leqslant 7, \tag{5.25}$$

or

$$j\leqslant k-5 \quad \text{and} \quad k\leqslant 15, \tag{5.26}$$

or

$$j\leqslant k-6 \quad \text{and} \quad k\leqslant 31. \tag{5.27}$$

We finish this section by indicating how to obtain new permissible exponents via the iterative schemes (M_1) and (M_2). We suppose in what follows that λ_r $(r\in\mathbb{N})$ are known permissible exponents, and we seek a new permissible exponent λ'_s.

(i) Process M_j^s. Consider first the iterative scheme (M_2) above. Suppose that the conditions of Lemma 5.7(I) hold with $u=s-j-1$. Then following the argument of §2 of [18] (see also §§11, 13), we find from Lemma 5.7(I) that λ'_s and $\boldsymbol{\phi}$ are determined by the relations

$$P\widetilde{H}_{j-1}\widetilde{M}_jQ_j^{\lambda_{s-j}}\approx P\widetilde{H}_j\widetilde{M}_jQ_j^{\lambda_{s-j}}(PM_1^{\varpi_j})^{-w_j}, \tag{5.28}$$

and the equations (4.2)-(4.4). Define $\mathcal{E}_i$ as in (4.5) for $1\leqslant i<j$, and define κ_i as in (4.6) for $(2\leqslant i\leqslant j)$. Also, define α_i, β_i, γ_i, δ_i for $1\leqslant i\leqslant j$ by

$$\alpha_j=k^{-1},\quad \beta_j=0,\quad \gamma_j=1-w_j,\quad \delta_j=w_j\varpi_j, \tag{5.29}$$

and for $i=j-1,\dots,2$ successively by

$$\begin{aligned}\delta_i&=\mathcal{E}_i+\kappa_{i+1}\alpha_{i+1}\delta_{i+1},\\ \gamma_i&=1+\mathcal{E}_i+\kappa_{i+1}\alpha_{i+1}\gamma_{i+1},\\ \beta_i&=\mathcal{E}_i+\kappa_{i+1}\alpha_{i+1}\beta_{i+1},\\ \alpha_i&=(2k+\beta_i)^{-1},\end{aligned} \tag{5.30}$$

and finally,

$$\begin{aligned}\delta_1&=\mathcal{E}_1+\kappa_2\alpha_2\delta_2,\\ \gamma_1&=1+\mathcal{E}_1+\kappa_2\alpha_2\gamma_2,\\ \beta_1&=0,\\ \alpha_1&=(2k+\delta_1)^{-1}.\end{aligned} \tag{5.31}$$

Then we find that $\boldsymbol{\phi}$ and λ'_s satisfy

$$\phi_i=\alpha_i(\gamma_i-\beta_i(\phi_2+\cdots+\phi_{i-1})-\delta_i\phi_1)\quad (2\leqslant i\leqslant j), \tag{5.32}$$

$$\phi_1 = \alpha_1\gamma_1 \tag{5.33}$$

and

$$\lambda_s' = \lambda_{s-1}(1-\phi_1) + 1 + (2s-2)\phi_1. \tag{5.34}$$

Thus, in a manner similar to that alluded to in §4, we are able to establish new permissible exponents by iterating this and allied procedures.

Suppose next that the conditions of Lemma 5.7(II) hold with $u = s - j - 1$. Then again following the argument of §2 of [18], we now find from Lemma 5.7(II) that λ_s' and $\boldsymbol{\phi}$ are determined by the relations

$$P\widetilde{H}_{j-1}\widetilde{M}_j Q_j^{\lambda_{s-j}} \approx P\widetilde{H}_j\widetilde{M}_j Q_j^{2s-2j-k}, \tag{5.35}$$

and the equations (4.2)-(4.4). Define $\mathcal{E}_i$ as in (4.5) for $1 \leqslant i < j$, and define κ_i as in (4.6) for $(2 \leqslant i \leqslant j)$. Also, define α_i, β_i, γ_i for $1 \leqslant i \leqslant j$ by

$$\alpha_j = \kappa_j^{-1}, \quad \beta_j = \kappa_j - k, \quad \gamma_j = 1 + \kappa_j - k, \tag{5.36}$$

and for $i = j-1, \ldots, 1$, successively by means of (4.7). Then we find that $\boldsymbol{\phi}$ and λ_s' satisfy (4.8)-(4.10), and again we are able to establish new permissible exponents by iterating this and related procedures.

(ii) Process N_j^s. Next consider the iterative scheme (M_1) above. Suppose that the conditions of Lemma 5.7(I') hold with $u = s - j - 1$. Then we find that λ_s' and $\boldsymbol{\phi}$ are determined by (5.28)-(5.34). Meanwhile, when the conditions of Lemma 5.7(II') hold with $u = s - j - 1$, one finds instead that λ_s' and $\boldsymbol{\phi}$ are determined by (5.35), (5.36) and (4.7)-(4.10). In either case we are able to establish new permissible exponents by iterating these and similar procedures.

6. Waring's problem

We defer announcing the permissible exponents obtained through our methods to §§9 to 22, but pause here to indicate how Theorem 1.1 may be established by means of the latter exponents. We require the following theorem, which is essentially a consequence of Corollary 1 to Theorem 4.2 of Wooley [22] and Theorem 5.1 of [22].

Theorem 6.1. *Let s, t and w be natural numbers satisfying $2s \geqslant k+1$, and suppose that Δ_n $(n = s,t,w)$ are admissible exponents. Define*

$$\sigma(k) = \frac{k - \Delta_t - \Delta_s\Delta_w}{2(s(k+\Delta_w-\Delta_t) + tw(1+\Delta_s))} \tag{6.1}$$

and

$$\lambda(k) = \frac{s(k-\Delta_t) + tw\Delta_s}{s(k+\Delta_w-\Delta_t) + tw(1+\Delta_s)}. \tag{6.2}$$

Suppose that

$$\tfrac{1}{2} < \lambda(k) < 1 - \sigma(k). \tag{6.3}$$

Then for each natural number v with $v \geqslant k$, and each admissible exponent Δ_v, we have

$$G(k) \leqslant \max \left\{ 2v+1+\left[\frac{\Delta_v}{\sigma(k)}\right], 4k \right\}. \tag{6.4}$$

We note that some minor modifications to the argument of the proof of Theorem 5.1 of [22] will be required in order to account for the use of the inequality (2.12) in place of the mean value

$$\int_0^1 |F_0(\alpha)^2 f_0(\alpha)^{2s}| d\alpha \ll P^{\lambda_{s+1}+\varepsilon}.$$

However, the replacement of the generating functions $f_0(\alpha)$ in the latter by $f_0^+(\alpha)$ in the former causes no technical problems, and affects only the singular integral in the asymptotic formula resulting from the application of the Hardy-Littlewood method. This singular integral, moreover, is easily bounded below by the expected quantity using only the methods of Chapter 2 of Vaughan [15], and so the desired conclusion follows with little difficulty.

In order to establish Theorem 1.1, one merely optimises the choice of $\sigma(k)$ through appropriate choices of s, t, w, and then one optimises the upper bound (6.4) for $G(k)$ through a suitable choice of v.

7. Distribution of αn^k modulo 1

We turn our attention now to the proofs of Theorems 1.2 and 1.3. Note first that the discussion of §6 of Wooley [22] leading to the proof of Theorem 1.2 of [22] establishes the following theorem.

Theorem 7.1. *Let $k \geqslant 4$, and suppose that $\sigma(k)$ and $\lambda(k)$ are defined as in* (6.1) *and* (6.2), *and satisfy* (6.3). *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Then there is a real number $N(\varepsilon, k)$ with the property that whenever $N \geqslant N(\varepsilon, k)$, one has*

$$\min_{1 \leqslant n \leqslant N} \|\alpha n^k\| \leqslant N^{\varepsilon - \sigma(k)}.$$

Thus the work expended in establishing Theorem 1.1 already yields the conclusion of Theorem 1.2.

The proof of Theorem 1.3 is a little more involved, though in principle this follows the argument of the proof of Theorem 1.1 of Wooley [20] in essentially all details.

Theorem 7.2. *Let k be a natural number with $7 \leqslant k \leqslant 20$, and let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Suppose that s is a natural number with $1 \leqslant s \leqslant k$, and that Δ_s is an admissible exponent derived through the methods described in* §§9–22. *Define*

$$\tau(k) = \frac{k - 2\Delta_s}{4s^2 - 1}.$$

Then there are infinitely many natural numbers n satisfying $\|\alpha n^k\| \leqslant n^{\varepsilon - \tau(k)}$.

Of course, the exponent claimed in the statement of Theorem 7.2 is valid in far greater generality, but our proof is much simplified by restricting to the methods of this paper. When P and H are large real numbers, denote by $U_s(P,H,R)$ the number of solutions of the diophantine equation

$$h_1x_1^k+\cdots+h_sx_s^k=g_1y_1^k+\cdots+g_sy_s^k \tag{7.1}$$

with

$$1\leqslant h_i,g_i\leqslant H \quad \text{and} \quad x_i,y_i\in\mathcal{A}(P,R) \quad (1\leqslant i\leqslant s).$$

In order to prove Theorem 7.2, we follow the argument of the proof of Lemma 4.2 of [20], together with the argument of the proof of Theorem 1.1 of that paper (described at the end of §4 of [20]). Thus we find that the conclusion of Theorem 7.2 will follow provided only that we establish that when s is a natural number with $1\leqslant s\leqslant k$, and λ_s is a permissible exponent, then

$$U_s(P,H,R)\ll H^{2s-1+\varepsilon}P^{\lambda_s+\varepsilon}. \tag{7.2}$$

A program for establishing such bounds is described in §3 of [20], but in light of subsequent developments we feel obliged to outline some of the necessary steps so far as the application at hand is concerned. Since a full account of such a proof would be costly in terms of space, we will be economical in the details by referring frequently to earlier work.

When $1\leqslant j\leqslant k$, we write

$$\mathfrak{f}_j(\alpha)=\sum_{1\leqslant g\leqslant H}f_j(g\alpha) \quad \text{and} \quad \mathfrak{F}_j(\alpha)=\sum_{1\leqslant g\leqslant H}F_j(g\alpha),$$

and we note that by orthogonality, one has

$$U_s(P,H,R)=\int_0^1|\mathfrak{f}_0(\alpha)|^{2s}d\alpha.$$

We will refer to an exponent λ_s as *derived* whenever the inequality (7.2) holds. Our aim is to show, at least when $1\leqslant s\leqslant k$, that the exponent λ_s is derived whenever λ_s is also a permissible exponent stemming from the methods described herein. In this context, we note that $\lambda_s=2s-1$ is always a derived exponent. For suppose that $\mathbf{g},\mathbf{h},\mathbf{x},\mathbf{y}$ is any solution of the equation (7.1) counted by $U_s(P,H,R)$. For each fixed choice of h_i,x_i $(2\leqslant i\leqslant s)$ and g_j,y_j $(1\leqslant j\leqslant s)$, an elementary estimate for the divisor function shows that there are at most $O((HP)^\varepsilon)$ possible choices for h_1 and x_1, whence

$$U_s(P,H,R)\ll(HP)^{2s-1+\varepsilon}.$$

Observe next that as a consequence of Lemma 3.4 of [20], and the argument of the proof of Lemma 3.1 of Wooley [19] (see, in particular, equation (3.7)), one has for $0\leqslant j\leqslant k-1$,

$$\begin{aligned}\int_0^1 &|\mathfrak{F}_j(\alpha)^2\mathfrak{f}_j(\alpha)^{2s}|d\alpha\\ &\ll P^\varepsilon H^{1+\varepsilon}\widetilde{H}_j\widetilde{M}_jM_{j+1}^{2s-1}\\ &\quad\times\Big(HP\widetilde{H}_j\widetilde{M}_{j+1}U_s(Q_j,H,R)+\int_0^1|\mathfrak{F}_{j+1}(\alpha)\mathfrak{f}_{j+1}(\alpha)^{2s}|d\alpha\Big).\end{aligned}$$

Thus, whenever (λ_s) is an existing sequence of derived exponents, one obtains the following analogues of Lemmata 2.1 and 2.2 above by essentially identical arguments.

Lemma 7.3. *We have*

$$\begin{aligned}\int_0^1 &|\mathfrak{F}_0(\alpha)^2\mathfrak{f}_0(\alpha)^{2s}|d\alpha\\ &\ll (HP)^\varepsilon M_1^{2s-1}\Big(H^{2s+1}PM_1Q_1^{\lambda_s}+H\int_0^1|\mathfrak{F}_1(\alpha)\mathfrak{f}_1(\alpha)^{2s}|d\alpha\Big).\end{aligned}$$

Lemma 7.4. *Whenever* $0<t<2s$ *and* $1\leqslant j\leqslant k-1$, *we have*

$$\int_0^1|\mathfrak{F}_j(\alpha)\mathfrak{f}_j(\alpha)^{2s}|d\alpha\ll(HP)^\varepsilon(H^{2t-1}Q_j^{\lambda_t})^{1/2}(H\widetilde{H}_j\widetilde{M}_jM_{j+1}^{4s-2t-1}T_{j+1})^{1/2},$$

where

$$T_{j+1}=P\widetilde{H}_j\widetilde{M}_{j+1}H^{4s-2t}Q_{j+1}^{\lambda_{2s-t}}+\int_0^1|\mathfrak{F}_{j+1}(\alpha)\mathfrak{f}_{j+1}(\alpha)^{4s-2t}|d\alpha.$$

The reader may wish to compare Lemma 7.4 with Lemma 3.5 of [20], which considers the special case with $s=t$.

Now observe from the tables in §§9-22 that for $7\leqslant k\leqslant 20$ and $1\leqslant s\leqslant k$, the iterative procedures described herein always terminate with processes of type $A_j^{s,l}$ or $B_{j,t}^{s,l}$ ($l=1,2$). A modicum of contemplation within the discussion of §4 above therefore leads one to the conclusion that the claimed bound (7.2) will follow, for any permissible exponent λ_s produced by the methods of this paper for $1\leqslant s\leqslant k$, so long as we are able to establish the estimates contained in the following lemmata.

Lemma 7.5. *With the hypotheses of the statement of Theorem* 3.4 *for* $1\leqslant l\leqslant k-2$, *subject to* (Ia) *or* (Ib), *one has*

$$\int_0^1|\mathfrak{F}_j(\alpha)|^{2^l}d\alpha\ll H^{2^l-1+\varepsilon}P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l-1}\widetilde{H}_j^{2^l-1},$$

subject to (Ic), *one has*

$$\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha \ll H^{2^l-1+\varepsilon}P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l-1+\sigma}\widetilde{H}_j^{2^l-1},$$

subject to (II), *one has*

$$\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha \ll H^{2^l-1+\varepsilon}P^{2^l-l-\frac{1}{3}+\varepsilon}\widetilde{M}_j^{2^l}\widetilde{H}_j^{2^l-1},$$

and subject to (III), *one has*

$$\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha \ll H^{2^l-1+\varepsilon}P^{2^l-l+\varepsilon}\widetilde{M}_j^{2^l}\widetilde{H}_j^{2^l-1}.$$

Lemma 7.6. *Suppose that* $2\leqslant l\leqslant k-2$. *Then one has*

$$\int_0^1 |\mathfrak{F}_{k-l}(\alpha)|^{2^l}\,d\alpha \ll H^{2^l-1+\varepsilon}P^{2^l-l+\varepsilon}\widetilde{M}_{k-l}^{2^l}\widetilde{H}_{k-l}^{2^l-1}.$$

In order to establish Lemmata 7.5 and 7.6, we note that in the diophantine equations underlying the mean values

$$\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha,$$

the equations differ from those underlying $R_j^{(s)}(P;\boldsymbol{\phi})$, defined in (3.1), only in so far as an additional linear variable in the interval $[1,H]$ occurs as a coefficient of each polynomial Ψ_j. Consequently, on following the argument of the proof of Lemma 3.1 above, we find that

$$\begin{aligned}\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^{l+1}}\,d\alpha \ll{}& P^{2^l-1}(H\widetilde{H}_j\widetilde{M}_j)^{2^l}\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha\\ &+P^{2^l-l-1}(H\widetilde{H}_j\widetilde{M}_j)^{2^l-1}\Big(S'\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^{l+1}}\,d\alpha\Big)^{1/2},\end{aligned}$$

where S' denotes the number of solutions of the equation

$$h\Psi_{j,l}(z;\mathbf{h};\mathbf{m};\mathbf{u}) = g\Psi_{j,l}(w;\mathbf{g};\mathbf{n};\mathbf{v}),$$

with the polynomials $\Psi_{j,l}$ as in the proof of Lemma 3.1, and with the variables in the same ranges, save that $1\leqslant g,h\leqslant H$. Then by a divisor estimate argument paralleling the start of the proof of Lemma 3.2 of [18], we find that

$$\begin{aligned}\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^{l+1}}\,d\alpha \ll{}& P^{2^l-1}(H\widetilde{H}_j\widetilde{M}_j)^{2^l}\int_0^1 |\mathfrak{F}_j(\alpha)|^{2^l}\,d\alpha\\ &+P^{2^{l+1}-2l-2}(H\widetilde{H}_j\widetilde{M}_j)^{2^{l+1}-2}H^{1+\varepsilon}R_{j+l}^{(1)}(P;\boldsymbol{\phi},\mathbf{0}).\end{aligned}$$

On considering the underlying diophantine equations, the bounds claimed in Lemmata 7.5 and 7.6 now follow by an inductive argument similar to that employed in the proofs of Theorems 3.4 and 3.5.

This completes the proof of Theorem 7.2.

8. Preliminary discussion of computations

By employing a computer to optimise the use of the methods described in §§2–5 of this paper, one derives an upper bound for a sequence (λ_s) of permissible exponents. In the tables presented in §§9–22, we record for each value of k the permissible exponents thus derived, together with the process yielding these exponents *towards the end* of the iteration process. Naturally, the conditions necessary for the application of the latter process may not initially hold. Under such circumstances, we begin by applying simpler, more robust, versions of such processes. Thus, for example, case (III) of Theorem 3.4 implies that processes A and B may always be applied with $\tau = 1$. In order to give some indication of the parameters $\boldsymbol{\phi}$ arising in these iterative processes, we record also the values of ϕ_1, ϕ_j, $\sum_{i=1}^{j} \phi_i$ (when $j > 1$), and when $j \geqslant 3$ we record also the value of

$$\phi_s^* = \max_{3 \leqslant I \leqslant j} \Big(\sum_{i=1}^{I} \phi_i + k(\phi_{I-1} + \phi_I) \Big),$$

corresponding to each process involving j differencing operations. Adjacent to the table, we discuss any issues pertaining to the applicability of iterative processes in the light of the conditions associated with the use of Theorems 3.4, 3.5 and Lemma 5.7. In particular, we note that the paramter $\chi_{j,l}$ is zero throughout unless otherwise indicated. Recorded values for λ_s and $\boldsymbol{\phi}$ are upper bounds, computations having been performed in double precision arithmetic.

Following the primary table, we record also the values of $\sigma(k)$, $\tau(k)$ and $G(k)$ (for $k \geqslant 9$) stemming from Theorems 6.1 and 7.2, the values of the former quantities recorded being lower bounds. We provide a parenthetic indication of the relevant parameters employed in the derivation of these values.

Note that the method of the proof of Theorem 5.1 of Wooley [22] shows that whenever Δ_v is an admissible exponent, then for

$$s \geqslant v + \left[\frac{\Delta_v}{2\sigma(k)} \right] + 1,$$

one has

$$\int_0^1 |f_0(\alpha)|^{2s} d\alpha \ll P^{2s-k},$$

whence $\lambda_s = 2s - k$ is a permissible exponent.

A final word is in order concerning the application of the processes M_j^s and N_j^s. When calculating a permissible exponent λ_s by means of Lemma 5.7, one frequently encounters conditions involving admissible exponents Δ_u with u substantially larger than s. Thus it is useful to prepare preliminary estimates by applying process M_j^s throughout, where j is sufficiently small that the condition (B) in the simple variants (5.25)-(5.27) is applicable. Since the conditions (D_1) and (D_2) are easy to check computationally, one obtains in this manner reasonably strong permissible exponents λ_s with s exceeding some suitable natural number s_0. Equipped with these preliminary bounds, we may subsequently refine the iterative procedures so as to attain the exponents claimed in the primary tables. Comments clarifying this process are included in each section.

We conclude by discussing a final simple process not without interest.

Process D^s. Suppose that $s\geqslant k$ and t is a natural number. Then whenever λ_s is a permissible exponent, then also the exponent λ'_{s+t} is permissible, where

$$\lambda'_{s+t} = \max\{\lambda_s + 2t(1-\sigma(k)), 2s+2t-k\},$$

and here $\sigma(k)$ is the exponent arising in the statement of Theorem 6.1. In order to establish this claim, we adapt the argument of the proof of Theorem 5.1 of Wooley [22]. Write

$$f(\alpha) = \sum_{x\in\mathcal{A}(P,R)} e(\alpha x^k) \quad \text{and} \quad g(\alpha) = \sum_{1\leqslant x\leqslant P} e(\alpha x^k).$$

Let $\mathfrak{m}$ denote the set of real numbers $\alpha \in [0,1)$ with the property that, whenever $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$ and $|q\alpha - a|\leqslant P^{1-k}$, one has $q > P$. Then as in the proof of Theorem 5.1 of [22], one has

$$\begin{aligned}\int_{\mathfrak{m}} |g(\alpha)^2 f(\alpha)^{2s+2t-2}|d\alpha &\ll \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\Big)^{2t} \int_0^1 |g(\alpha)^2 f(\alpha)^{2s-2}|d\alpha \\ &\ll (P^{1-\sigma(k)+\varepsilon})^{2t} P^{\lambda_s+\varepsilon}.\end{aligned}$$

On the other hand, provided that $s\geqslant k$, one may apply a standard pruning argument, of the type described in §5 of Vaughan [13], to show that

$$\int_{\mathfrak{M}} |g(\alpha)^2 f(\alpha)^{2s+2t-2}|d\alpha \ll P^{2s+2t-k}.$$

By considering the underlying diophantine equations, the claimed conclusion follows on noting that

$$\int_0^1 |f(\alpha)|^{2s+2t} d\alpha \ll \int_0^1 |g(\alpha)^2 f(\alpha)^{2s+2t-2}|d\alpha.$$

As a consequence of the process D^s, we may restrict attention to those s for which the processes A, B, N or M demonstrate that the permissible exponent λ_s satisfies

$$\lambda_s < \lambda_{s-1} + 2(1-\sigma(k)),$$

for all permissible exponents λ_{s-1}.

9. Permissible exponents for seventh powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table for $3\leqslant s\leqslant 12$ are identical with those on p.237 of [18], and indeed for $s = 3, 4$, these exponents were established earlier by Vaughan [14]. We remark that in this range of s, one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ throughout (see [18] for details). Our computations for $s\geqslant 13$ depend on first obtaining preliminary estimates by applying the process M_2^s throughout (noting (5.25) and checking (D_1) or (D_2)). In this way we obtain the preliminary permissible exponents

$$\lambda_{13} = 19.211, \quad \lambda_{14} = 21.127, \quad \lambda_{15} = 23.073, \quad \lambda_{16} = 25.019,$$

and $\lambda_s = 2s - 7$ for $s\geqslant 17$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 13, 14$. With process M_4^s, one finds that Lemma 5.7(I) holds with $u = s - 5$ by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(b) $s = 15$. With process M_3^{15}, one finds that Lemma 5.7(I) holds with $u = 11$ by virtue of conditions (D_1), (B).

(c) $s = 16$. With process M_2^{16}, one finds that Lemma 5.7(I) holds with $u = 13$ by virtue of conditions (D_1), (B).

(d) $s\geqslant 17$. One finds that process D^s applies.

Table of permissible exponents for $\mathbf{k = 7}$

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0639191	0.03195955			
4	$A_1^{4,2}$	4.2641175	0.06818557			
5	$B_{2,5}^{5,2}$	5.5891167	0.08699398	0.0355	0.1225	
6	$A_2^{6,2}$	7.0143820	0.09641272	0.0694	0.1658	
7	$B_{3,6}^{7,2}$	8.5410894	0.10564538	0.0406	0.2343	1.1347
8	$A_3^{8,2}$	10.1526323	0.11202654	0.0691	0.2800	1.4554
9	$B_{4,8}^{9,2}$	11.8469485	0.11873997	0.0416	0.3577	1.6983
10	$A_4^{10,2}$	13.6055676	0.12329153	0.0661	0.4030	1.8315
11	$A_4^{11,2}$	15.4242973	0.12803790	0.0859	0.4429	1.9600
12	$A_4^{12,2}$	17.2932208	0.13214156	0.1027	0.4781	2.0785
13	M_4^{13}	19.1925374	0.13409068	0.1072	0.4919	2.1387
14	M_4^{14}	21.1139297	0.13535033	0.1072	0.4957	2.1606
15	M_3^{15}	23.0528848	0.13635572	0.1226	0.3903	2.1673
16	M_2^{16}	25.0105382	0.13784908	0.1327	0.2706	
$s \geqslant 17$	D^s	$2s-7$				

$$\sigma(7) = 0.017475 \quad (s = 14,\, t = 5,\, w = 8),$$

$$\tau(7) = 0.020777 \quad (s = 6).$$

10. Permissible exponents for eighth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those provided by [16, 18] for $3 \leqslant s \leqslant 20$. Broadly speaking one can follow the discussion of [18] for $3 \leqslant s \leqslant 14$, though the improvements contained in Theorem 3.4 lead to sharper estimates, and permit a slightly more powerful iterative process. In particular, one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ for $3 \leqslant s \leqslant 12$, and $\tau_{5,2} = 0.002$ for $s = 13, 14$ (see §11 of [18] for details). Our computations for $s \geqslant 15$ depend on first obtaining preliminary estimates by applying the process M_3^s for $16 \leqslant s \leqslant 20$, and M_2^s for $s \geqslant 21$ (noting (5.26) and checking (D_1) or (D_2)). In this way we obtain the preliminary permissible exponents

$$\lambda_{15} = 22.282, \quad \lambda_{16} = 24.206, \quad \lambda_{17} = 26.143, \quad \lambda_{18} = 28.098,$$

$$\lambda_{19} = 30.061, \quad \lambda_{20} = 32.031, \quad \lambda_{21} = 34.010,$$

and $\lambda_s = 2s - 8$ for $s \geqslant 22$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $15\leqslant s\leqslant 17$. With process M_5^s, one finds that Lemma 5.7(I) holds with $u=s-6$ by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(b) $s=18,19$. With process M_4^s, one finds that Lemma 5.7(I) holds with $u=s-5$ by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s\geqslant 20$. One finds that process D^s applies.

Table of permissible exponents for k = 8

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0469787	0.02348931			
4	$A_1^{4,2}$	4.2164889	0.05740233			
5	$B_{2,6}^{5,2}$	5.4915710	0.07270549	0.0222	0.0949	
6	$A_2^{6,2}$	6.8566563	0.08097841	0.0563	0.1373	
7	$B_{3,7}^{7,2}$	8.3105992	0.08825831	0.0264	0.1881	0.9864
8	$A_3^{8,2}$	9.8428621	0.09355343	0.0559	0.2322	1.3410
9	$A_3^{9,2}$	11.4529104	0.09907986	0.0750	0.2646	1.5885
10	$A_4^{10,2}$	13.1283069	0.10315980	0.0550	0.3379	1.7214
11	$A_4^{11,2}$	14.8664781	0.10742204	0.0741	0.3750	1.8485
12	$A_5^{12,2}$	16.6561197	0.11069450	0.0528	0.4510	1.9401
13	$A_5^{13,2}$	18.4901012	0.11356143	0.0686	0.4849	2.0172
14	$A_5^{14,2}$	20.3623532	0.11614698	0.0820	0.5156	2.0883
15	M_5^{15}	22.2661078	0.11832893	0.0938	0.5428	2.1728
16	M_5^{16}	24.1891161	0.11934590	0.0938	0.5484	2.2022
17	M_5^{17}	26.1294925	0.12039309	0.0938	0.5527	2.2249
18	M_4^{18}	28.0833353	0.12119204	0.1075	0.4620	2.2385
19	M_4^{19}	30.0473193	0.12176644	0.1075	0.4638	2.2447
20	D^{20}	32.0186056				
$s\geqslant 21$	D^s	$2s-8$				

$$\sigma(8)=0.014356 \quad (s=16,\ t=6,\ w=10),$$

$$\tau(8)=0.017327 \quad (s=7).$$

11. Permissible exponents for ninth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve those provided by [18] for $5\leqslant s\leqslant 25$. For $s=3,4$, the exponents recorded in the table were established earlier by Vaughan [14]. Broadly speaking, we may again follow the discussion of [18] for $5\leqslant s\leqslant 17$, though the improvements contained in Theorem 3.4 lead to sharper estimates, and permit slightly more powerful processes. We note in particular that when $k\geqslant 9$ and $1\leqslant j\leqslant 3$, then in view of the inequality $0\leqslant\phi_i\leqslant 1/k$ $(1\leqslant i\leqslant j)$, one has

$$\phi_1+\cdots+\phi_j\leqslant\tfrac{3}{k}\leqslant\tfrac{1}{3}. \tag{11.1}$$

Thus, for $1\leqslant j\leqslant 3$, the condition (δ) of Theorem 3.4 is automatically satisfied. Combining the latter observation with the methods of [18], it follows that one may take $\tau_{j,l}=0$ $(l=1,2)$ for $3\leqslant s\leqslant 11$ (in which interval our methods make use of a choice of j with $1\leqslant j\leqslant 4$). When $j=5$ and 6, it follows from Theorem 3.4(Ib) case (iii) that one may take $\tau_{j,l}=0$ $(l=1,2)$ provided only that

$$\sum_{i=1}^{I}\phi_i+9(\phi_{I-1}+\phi_I)\leqslant 2,$$

when $I=3,4,5$. The computational verification of this inequality leads to the conclusion that one may take $\tau_{j,l}=0$ $(l=1,2)$ also for $12\leqslant s\leqslant 14$. Finally, when $j=6$ and $15\leqslant s\leqslant 17$, it follows as in §11 of [18] that one may take $\tau_{j,l}=0.002565$ $(l=1,2)$ whenever $\phi_1\leqslant 0.107131$.

As in the previous cases, our computations for $s\geqslant 18$ depend on first obtaining preliminary estimates by applying the process M_4^s for $18\leqslant s\leqslant 23$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s\geqslant 24$. In this way we obtain the preliminary permissible exponents

$$\lambda_{18}=27.260,\quad \lambda_{19}=29.199,\quad \lambda_{20}=31.150,$$

$$\lambda_{21}=33.120,\quad \lambda_{22}=35.080,\quad \lambda_{23}=37.055,$$

and by virtue of the preliminary exponent

$$\sigma(9)=0.01212\quad (s=19,\ t=6,\ w=12),$$

we have also

$$\lambda_s=\max\{2s-9,\ 37.055+2(s-23)(1-0.01212)\}$$

for $s>23$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s=18,19$. With process M_6^s, one finds that Lemma 5.7(I) holds with $u=s-7$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(b) $20\leqslant s\leqslant 22$. With process M_5^s, one finds that Lemma 5.7(I) holds with $u=s-6$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s\geqslant 23$. One finds that process D^s applies.

Table of permissible exponents for k = 9

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0358052	0.01790259			
4	$A_1^{4,2}$	4.1822894	0.04941786			
5	$B_{2,6}^{5,2}$	5.4197057	0.06218814	0.0116	0.0737	
6	$B_{2,6}^{6,2}$	6.7383084	0.06955946	0.0463	0.1158	
7	$A_2^{7,2}$	8.1356346	0.07551302	0.0626	0.1381	
8	$B_{3,7}^{8,2}$	9.6039271	0.07985390	0.0450	0.1956	1.2367
9	$A_3^{9,2}$	11.1425026	0.08420410	0.0620	0.2231	1.4729
10	$A_4^{10,2}$	12.7463430	0.08805551	0.0451	0.2870	1.6267
11	$A_4^{11,2}$	14.4105835	0.09157319	0.0625	0.3194	1.7451
12	$A_5^{12,2}$	16.1292111	0.09468812	0.0451	0.3870	1.8387
13	$A_5^{13,2}$	17.8959526	0.09741610	0.0607	0.4194	1.9155
14	$B_{6,11}^{14,2}$	19.7055987	0.09990639	0.0420	0.4882	1.9858
15	$A_6^{15,2}$	21.5507274	0.10189148	0.0553	0.5185	2.0414
16	$A_6^{16,2}$	23.4269614	0.10370526	0.0673	0.5468	2.1092
17	$A_6^{17,2}$	25.3292029	0.10524175	0.0775	0.5717	2.1717
18	M_6^{18}	27.2520471	0.10643130	0.0834	0.5885	2.2317
19	M_6^{19}	29.1901860	0.10724097	0.0834	0.5937	2.2558
20	M_5^{20}	31.1420569	0.10804665	0.0958	0.5163	2.2898
21	M_5^{21}	33.1033373	0.10852186	0.0958	0.5185	2.2962
22	M_5^{22}	35.0727119	0.10895936	0.0958	0.5203	2.3010
$s \geqslant 23$	D^s					

$$\sigma(9) = 0.012183 \quad (s = 19,\, t = 6,\, w = 12),$$

$$\tau(9) = 0.014871 \quad (s = 8), \quad G(9) \leqslant 50 \quad (v = 22).$$

12. Permissible exponents for tenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. Our strategy is similar to that described in previous sections. Note first that for $3 \leqslant s \leqslant 11$, the condition ($\delta$) of Theorem 3.4 is satisfied. Since when $j = 5, 6$, one of the conditions (α) and (β) of Theorem 3.4 is satisfied, and one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied for $3 \leqslant s \leqslant 15$, we deduce that one may take $\tau_{j,l} = 0$ ($l = 1, 2$) for $3 \leqslant s \leqslant 15$.

In order to discuss permissible exponents for $16 \leqslant s \leqslant 20$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §22 that when $j = 6$ or 7 and $8 \leqslant j + l \leqslant 9$, one has $J = 1$, and hence one may take

$$\delta_3 = 0.0035377, \quad \delta_4 = 0.0372112, \quad \delta_6 = 0.2457501,$$

$$\delta_7 = 0.4042791 \quad \text{and} \quad \delta_8 = 0.5946271$$

(note that §22 is independent of §§9–21). Thus we deduce that one may take

$$\tau_{6,l} = \tfrac{1}{3}\delta_3 < 0.00118 \quad \text{whenever} \quad \phi_1 \leqslant 0.097668 \quad (l = 2, 3),$$

and that one may take

$$\tau_{7,l} = \tfrac{1}{7}(\delta_3 + \delta_4) < 0.00583 \quad \text{whenever} \quad \phi_1 \leqslant 0.095055 \quad (l = 2),$$

and otherwise, one may take

$$\tau_{7,l} = \tfrac{1}{7}\delta_7 < 0.05776.$$

As in the previous cases, our computations for $s \geqslant 21$ depend on first obtaining preliminary estimates by applying the process M_5^s for $21 \leqslant s \leqslant 26$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s \geqslant 27$. In this way we obtain the preliminary permissible exponents

$$\lambda_{21} = 32.249, \quad \lambda_{22} = 34.198, \quad \lambda_{23} = 36.156,$$

$$\lambda_{24} = 38.122, \quad \lambda_{25} = 40.094, \quad \lambda_{26} = 42.072,$$

and by virtue of the preliminary exponent

$$\sigma(10) = 0.01054 \quad (s = 22,\ t = 7,\ w = 13),$$

we have also

$$\lambda_s = \max\{2s - 10,\ 42.072 + 2(s - 26)(1 - 0.01054)\}$$

for $s > 26$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 21, 22$. With process N_7^s, one finds that Lemma 5.7(I') holds with $u = s - 8$, by virtue of condition (A_1).

(b) $23 \leqslant s \leqslant 25$. With process M_6^s, one finds that Lemma 5.7(I) holds with $u = s - 7$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s = 26$. With process M_6^s, one finds that Lemma 5.7(II) holds with $u = 19$, by virtue of conditions (D_2), (C_1), (C_4), (C_5).

(d) $s \geqslant 27$. One finds that process D^s applies.

Table of permissible exponents for k = 10

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0281105	0.01405522			
4	$A_1^{4,2}$	4.1568114	0.04330611			
5	$B_{2,6}^{5,2}$	5.3659530	0.05441877	0.0042	0.0586	
6	$B_{2,6}^{6,2}$	6.6465971	0.06056135	0.0375	0.0981	
7	$A_2^{7,2}$	7.9967162	0.06540123	0.0525	0.1179	
8	$B_{3,7}^{8,2}$	9.4143540	0.06956822	0.0361	0.1670	1.1412
9	$A_3^{9,2}$	10.8945712	0.07291878	0.0516	0.1912	1.3731
10	$A_4^{10,2}$	12.4375675	0.07641992	0.0368	0.2472	1.5499
11	$A_4^{11,2}$	14.0371956	0.07929039	0.0525	0.2751	1.6556
12	$A_5^{12,2}$	15.6914013	0.08215771	0.0377	0.3350	1.7513
13	$A_5^{13,2}$	17.3943657	0.08460686	0.0524	0.3645	1.8239
14	$B_{6,11}^{14,2}$	19.1426918	0.08695770	0.0366	0.4265	1.8937
15	$A_6^{15,2}$	20.9303709	0.08892986	0.0502	0.4559	1.9503
16	$B_{6,13}^{16,3}$	22.7537459	0.09078376	0.0627	0.4843	2.0175
17	$A_7^{17,2}$	24.6071999	0.09230268	0.0456	0.5472	2.0724
18	$A_7^{18,2}$	26.4867878	0.09364492	0.0558	0.5724	2.1251
19	$A_7^{19,2}$	28.3886784	0.09480400	0.0647	0.5951	2.1845
20	$A_7^{20,2}$	30.3094873	0.09580462	0.0734	0.6171	2.2431
21	N_7^{21}	32.2449884	0.09653784	0.0750	0.6258	2.2800
22	N_7^{22}	34.1926960	0.09715085	0.0750	0.6306	2.3008
23	M_6^{23}	36.1509648	0.09770971	0.0863	0.5616	2.3354
24	M_6^{24}	38.1169804	0.09808226	0.0863	0.5639	2.3420
25	M_6^{25}	40.0895832	0.09841150	0.0863	0.5658	2.3469
26	M_6^{26}	42.0677228	0.09869813	0.0867	0.5681	2.3577
$s \geqslant 27$	D^s					

$$\sigma(10) = 0.010569 \quad (s = 22,\ t = 7,\ w = 13),$$

$$\tau(10) = 0.013036 \quad (s = 9), \quad G(10) \leqslant 59 \quad (v = 26).$$

13. Permissible exponents for eleventh powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3 \leqslant s \leqslant 13$, the condition (δ) of Theorem 3.4 is satisfied. Since when $j = 6, 7$, one of the conditions (α), (β) and (γ) of Theorem 3.4 is satisfied, and one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied for $3 \leqslant s \leqslant 17$, we deduce that one may take $\tau_{j,l} = 0$ $(l = 1, 2, 3)$ for $3 \leqslant s \leqslant 17$.

In order to discuss permissible exponents for $18\leqslant s\leqslant 22$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j=7$ or 8 and $l=2$, one has $J=1$, and hence one may take

$$\delta_3=0.0025439,\quad \delta_4=0.0292912,\quad \delta_8=0.5257736.$$

Thus we deduce that one may take

$$\tau_{7,l}=\tfrac{1}{7}(\delta_3+\delta_4)<0.00455\quad\text{whenever}\quad \phi_1\leqslant 0.087205\quad (l=2),$$

and that one may take

$$\tau_{8,l}=\tfrac{1}{4}\delta_4<0.00733\quad\text{whenever}\quad \phi_1\leqslant 0.087205\quad (l=2).$$

As in the previous cases, our computations for $s\geqslant 23$ depend on first obtaining preliminary estimates by applying the process M_6^s for $23\leqslant s\leqslant 29$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s\geqslant 30$. In this way we obtain the preliminary permissible exponents

$$\lambda_{23}=35.299,\quad \lambda_{24}=37.244,\quad \lambda_{25}=39.199,\quad \lambda_{26}=41.161,$$

$$\lambda_{27}=43.130,\quad \lambda_{28}=45.105,\quad \lambda_{29}=47.084,$$

and by virtue of the preliminary exponent

$$\sigma(11)=0.00930\quad (s=25,\ t=7,\ w=15),$$

we have also

$$\lambda_s=\max\{2s-11,\ 47.084+2(s-29)(1-0.00930)\}$$

for $s>29$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s=23,24$*.* With process N_8^s, one finds that Lemma 5.7(I') holds with $u=s-9$, by virtue of condition (A_1).

(b) $25\leqslant s\leqslant 28$*.* With process M_7^s, one finds that Lemma 5.7(I) holds with $u=s-8$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s=29$*.* With process N_6^s, one finds that Lemma 5.7(I') holds with $u=s-7$, by virtue of condition (A_1).

(d) $s\geqslant 30$*.* One finds that process D^s applies.

Table of permissible exponents for k = 11

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0221905	0.01109521			
4	$B_{1,5}^{4,2}$	4.1346363	0.03776126			
5	$A_1^{5,2}$	5.3216133	0.04837243			
6	$B_{2,6}^{6,2}$	6.5727501	0.05368021	0.0312	0.0848	
7	$A_2^{7,2}$	7.8855603	0.05763697	0.0449	0.1026	
8	$B_{3,8}^{8,2}$	9.2614256	0.06147175	0.0287	0.1442	1.0534
9	$A_3^{9,2}$	10.6937742	0.06416025	0.0434	0.1662	1.2881
10	$B_{4,9}^{10,2}$	12.1849727	0.06723014	0.0289	0.2140	1.4815
11	$A_4^{11,2}$	13.7292224	0.06964143	0.0441	0.2396	1.5798
12	$B_{5,10}^{12,2}$	15.3262982	0.07219102	0.0301	0.2917	1.6759
13	$A_5^{13,2}$	16.9712740	0.07435992	0.0446	0.3191	1.7442
14	$B_{6,11}^{14,2}$	18.6621448	0.07651920	0.0307	0.3745	1.8126
15	$A_6^{15,2}$	20.3940119	0.07837637	0.0441	0.4022	1.8679
16	$B_{6,13}^{16,3}$	22.1640483	0.08016212	0.0566	0.4295	1.9304
17	$A_7^{17,2}$	23.9674841	0.08168359	0.0422	0.4874	1.9847
18	$A_7^{18,2}$	25.8009828	0.08307973	0.0533	0.5133	2.0398
19	$A_8^{19,2}$	27.6607360	0.08429767	0.0378	0.5710	2.0979
20	$A_8^{20,2}$	29.5431019	0.08534127	0.0466	0.5939	2.1448
21	$A_8^{21,2}$	31.4450976	0.08625844	0.0548	0.6151	2.1947
22	$A_8^{22,2}$	33.3638548	0.08704554	0.0617	0.6340	2.2473
23	N_8^{23}	35.2968576	0.08772003	0.0682	0.6516	2.2983
24	N_8^{24}	37.2413126	0.08824091	0.0682	0.6568	2.3214
25	M_7^{25}	39.1958837	0.08872562	0.0786	0.5959	2.3658
26	M_7^{26}	41.1582991	0.08907859	0.0786	0.5988	2.3733
27	M_7^{27}	43.1274069	0.08938707	0.0786	0.6012	2.3800
28	M_7^{28}	45.1020502	0.08964222	0.0786	0.6031	2.3893
29	N_6^{29}	47.0818525	0.08990704	0.0848	0.5283	2.4164
$s\geqslant 30$	D^s					

$$\sigma(11)=0.009322 \quad (s=25,\ t=7,\ w=15),$$

$$\tau(11)=0.011604 \quad (s=10), \quad G(11)\leqslant 67 \quad (v=29).$$

14. Permissible exponents for twelfth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s\geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3\leqslant s\leqslant 14$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ for $3 \leqslant s \leqslant 14$. When $j = 6$ and $s = 15, 16$, meanwhile, we must resort to Theorem 3.4(II)(1). Here we note that condition (iii) is satisfied, and thus the estimate (4.1) holds for $j = 6$ and $l = 2, 3$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Next, when $j = 7$ and $s = 17, 18$, we may apply case (β) of Theorem 3.4(Ib) in combination with the condition (iii) to deduce that one may take $\tau_{7,2} = 0$.

In order to discuss permissible exponents for $19 \leqslant s \leqslant 25$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 8$ or 9 and $10 \leqslant j + l \leqslant 11$, one has $J = 1$, and hence one may take

$$\delta_4 = 0.0234059, \quad \delta_5 = 0.0866022, \quad \delta_8 = 0.4689321,$$

$$\delta_9 = 0.6501924, \quad \delta_{10} = 0.8586937.$$

Thus we deduce that one may take

$$\tau_{8,l} = \tfrac{1}{4}\delta_4 < 0.00586 \quad \text{whenever} \quad \phi_1 \leqslant 0.080502 \quad (l = 2, 3),$$

and that one may take

$$\tau_{9,l} = \tfrac{1}{9}(\delta_4 + \delta_5) < 0.01223 \quad \text{whenever} \quad \phi_1 \leqslant 0.078831 \quad (l = 2),$$

and otherwise, one may take

$$\tau_{9,l} = \tfrac{1}{9}\delta_9 < 0.07225 \quad (l = 2).$$

As in the previous cases, our computations for $s \geqslant 26$ depend on first obtaining preliminary estimates by applying the process M_7^s for $26 \leqslant s \leqslant 33$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s \geqslant 34$. In this way we obtain the preliminary permissible exponents

$$\lambda_{26} = 40.290, \quad \lambda_{27} = 42.241, \quad \lambda_{28} = 44.200, \quad \lambda_{29} = 46.166,$$

$$\lambda_{30} = 48.138, \quad \lambda_{31} = 50.114, \quad \lambda_{32} = 52.094, \quad \lambda_{33} = 54.077,$$

and by virtue of the preliminary exponent

$$\sigma(12) = 0.00834 \quad (s = 28,\ t = 8,\ w = 17),$$

we have also

$$\lambda_s = \max\{2s - 12,\ 54.077 + 2(s - 33)(1 - 0.00834)\}$$

for $s > 33$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 26, 27$. With process N_9^s, one finds that Lemma 5.7(I') holds with $u = s - 10$, by virtue of condition (A_1).

(b) $28 \leqslant s \leqslant 32$. With process M_8^s, one finds that Lemma 5.7(I) holds with $u = s - 9$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s \geqslant 33$. One finds that process D^s applies.

Table of permissible exponents for k = 12

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0173811	0.00869053			
4	$B_{1,5}^{4,2}$	4.1139674	0.03238304			
5	$A_1^{5,2}$	5.2806367	0.04288933			
6	$B_{2,7}^{6,2}$	6.5078011	0.04813455	0.0252	0.0733	
7	$A_2^{7,2}$	7.7910496	0.05157287	0.0392	0.0907	
8	$B_{3,8}^{8,2}$	9.1322748	0.05495699	0.0221	0.1252	0.9671
9	$A_3^{9,2}$	10.5253191	0.05723065	0.0371	0.1466	1.2185
10	$B_{4,9}^{10,2}$	11.9729266	0.05988316	0.0222	0.1871	1.4243
11	$A_4^{11,2}$	13.4700805	0.06193465	0.0373	0.2112	1.5174
12	$B_{5,10}^{12,2}$	15.0174771	0.06417372	0.0238	0.2566	1.6127
13	$A_5^{13,2}$	16.6110110	0.06607653	0.0378	0.2819	1.6761
14	$A_5^{14,2}$	18.2496682	0.06802194	0.0503	0.3060	1.7421
15	$A_6^{15,2}$	19.9296021	0.06973444	0.0387	0.3580	1.7957
16	$B_{6,13}^{16,3}$	21.6486622	0.07140335	0.0515	0.3844	1.8541
17	$A_7^{17,2}$	23.4028589	0.07285983	0.0378	0.4355	1.9056
18	$A_7^{18,2}$	25.1895563	0.07423677	0.0490	0.4606	1.9583
19	$A_8^{19,2}$	27.0053277	0.07546142	0.0357	0.5142	2.0143
20	$A_8^{20,2}$	28.8470137	0.07655400	0.0453	0.5376	2.0630
21	$B_{8,17}^{21,3}$	30.7117485	0.07753392	0.0540	0.5596	2.1187
22	$A_9^{22,2}$	32.5965148	0.07837940	0.0396	0.6128	2.1644
23	$A_9^{23,2}$	34.4988383	0.07912700	0.0477	0.6340	2.2080
24	$A_9^{24,2}$	36.4163328	0.07977408	0.0540	0.6516	2.2565
25	$A_9^{25,2}$	38.3468951	0.08033400	0.0596	0.6676	2.3037
26	N_9^{26}	40.2885464	0.08080691	0.0625	0.6782	2.3393
27	N_9^{27}	42.2395410	0.08120210	0.0625	0.6830	2.3609
28	M_8^{28}	44.1986746	0.08155581	0.0721	0.6272	2.3983
29	M_8^{29}	46.1643984	0.08183181	0.0721	0.6300	2.4054
30	M_8^{30}	48.1357634	0.08207146	0.0721	0.6323	2.4160
31	M_8^{31}	50.1118679	0.08227285	0.0721	0.6343	2.4251
32	M_8^{32}	52.0919461	0.08244173	0.0721	0.6360	2.4327
$s \geqslant 33$	D^s					

$$\sigma(12) = 0.008349 \quad (s = 28,\, t = 8,\, w = 17),$$
$$\tau(12) = 0.010475 \quad (s = 11), \quad G(12) \leqslant 76 \quad (v = 32).$$

15. Permissible exponents for thirteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3 \leqslant s \leqslant 15$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ for $3 \leqslant s \leqslant 15$. When $j = 6$ or 7 and $s = 16, 17, 18$, meanwhile, we must resort to Theorem 3.4(II)(1). Here we note that condition (iii) is satisfied, and thus the estimate (4.1) holds for $j = 6$ and $l = 2, 3$, and likewise for $j = 7$ and $l = 2$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Next, when $j = 8$ and $s = 19, 20$, we may apply case (β) of Theorem 3.4(Ib) in combination with the condition (iii) to deduce that one may take $\tau_{8,2} = 0$.

In order to discuss permissible exponents for $21 \leqslant s \leqslant 28$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 9$ or 10 and $11 \leqslant j + l \leqslant 12$, one has $J = 1$, and hence one may take

$$\delta_4 = 0.0190100, \quad \delta_5 = 0.0738636, \quad \delta_8 = 0.4222375, \quad \delta_{10} = 0.7792591.$$

Thus we deduce that one may take

$$\tau_{9,l} = \tfrac{1}{9}(\delta_4 + \delta_5) < 0.01032 \quad \text{whenever} \quad \phi_1 \leqslant 0.073360 \quad (l = 2, 3),$$

and that one may take

$$\tau_{10,l} = \tfrac{1}{5}\delta_5 < 0.01478 \quad \text{whenever} \quad \phi_1 \leqslant 0.073360 \quad (l = 2),$$

and otherwise, one may take

$$\tau_{10,l} = \tfrac{1}{10}\delta_{10} < 0.07793 \quad (l = 2).$$

As in the previous cases, our computations for $s \geqslant 29$ depend on first obtaining preliminary estimates by applying the process M_8^s for $29 \leqslant s \leqslant 36$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s \geqslant 37$. In this way we obtain the preliminary permissible exponents

$$\lambda_{29} = 45.284, \quad \lambda_{30} = 47.240, \quad \lambda_{31} = 49.203, \quad \lambda_{32} = 51.171,$$
$$\lambda_{33} = 53.144, \quad \lambda_{34} = 55.122, \quad \lambda_{35} = 57.102, \quad \lambda_{36} = 59.086,$$

and by virtue of the preliminary exponent

$$\sigma(13) = 0.00755 \quad (s = 31,\, t = 9,\, w = 18),$$

we have also

$$\lambda_s = \max\{2s - 13,\ 59.086 + 2(s - 36)(1 - 0.00755)\}$$

for $s > 36$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

Table of permissible exponents for k = 13

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0139128	0.00695637			
4	$B_{1,5}^{4,2}$	4.0980713	0.02818356			
5	$A_1^{5,2}$	5.2480334	0.03843282			
6	$B_{2,7}^{6,2}$	6.4550068	0.04355532	0.0200	0.0636	
7	$B_{2,7}^{7,2}$	7.7136017	0.04663575	0.0344	0.0811	
8	$B_{3,9}^{8,2}$	9.0257224	0.04965016	0.0166	0.1095	0.8876
9	$A_3^{9,2}$	10.3855801	0.05159785	0.0320	0.1307	1.1586
10	$A_3^{10,2}$	11.7953083	0.05380951	0.0428	0.1471	1.3592
11	$A_4^{11,2}$	13.2521623	0.05568206	0.0317	0.1880	1.4644
12	$A_4^{12,2}$	14.7557170	0.05756333	0.0429	0.2072	1.5517
13	$A_5^{13,2}$	16.3039366	0.05930365	0.0319	0.2510	1.6178
14	$A_5^{14,2}$	17.8953488	0.06099508	0.0433	0.2724	1.6782
15	$A_6^{15,2}$	19.5276970	0.06257992	0.0324	0.3188	1.7316
16	$B_{6,13}^{16,3}$	21.1988910	0.06409230	0.0444	0.3427	1.7823
17	$A_7^{17,2}$	22.9062626	0.06549065	0.0345	0.3928	1.8359
18	$A_7^{18,2}$	24.6473038	0.06679816	0.0460	0.4176	1.8877
19	$A_8^{19,2}$	26.4190897	0.06798261	0.0323	0.4638	1.9380
20	$A_8^{20,2}$	28.2190701	0.06907751	0.0421	0.4868	1.9866
21	$A_9^{21,2}$	30.0445818	0.07007187	0.0302	0.5371	2.0395
22	$A_9^{22,2}$	31.8929000	0.07095680	0.0390	0.5591	2.0860
23	$B_{9,18}^{23,3}$	33.7616279	0.07175359	0.0470	0.5797	2.1319
24	$A_{10}^{24,2}$	35.6483904	0.07245756	0.0336	0.6289	2.1808
25	$A_{10}^{25,2}$	37.5510006	0.07307633	0.0401	0.6470	2.2212
26	$A_{10}^{26,2}$	39.4675269	0.07362249	0.0470	0.6654	2.2610
27	$A_{10}^{27,2}$	41.3961543	0.07409771	0.0521	0.6805	2.3067
28	$A_{10}^{28,2}$	43.3352806	0.07451110	0.0566	0.6939	2.3485
29	N_{10}^{29}	45.2834077	0.07486365	0.0577	0.7010	2.3754
30	N_{10}^{30}	47.2392765	0.07516706	0.0577	0.7053	2.3948
31	M_9^{31}	49.2018815	0.07543499	0.0666	0.6538	2.4262
32	M_9^{32}	51.1701090	0.07565390	0.0666	0.6565	2.4365
33	M_9^{33}	53.1431803	0.07584409	0.0666	0.6588	2.4467
34	M_9^{34}	55.1203776	0.07600614	0.0666	0.6607	2.4555
35	M_9^{35}	57.1010835	0.07614400	0.0666	0.6624	2.4629
36	N_8^{36}	59.0849135	0.07627230	0.0718	0.5990	2.4879
$s \geqslant 37$	D^s					

$$\sigma(13) = 0.007556 \quad (s = 31,\ t = 9,\ w = 18),$$

$$\tau(13) = 0.009545 \quad (s = 12), \quad G(13) \leqslant 84 \quad (v = 36).$$

(a) $s=29,30$. With process N_{10}^s, one finds that Lemma 5.7(I') holds with $u=s-11$, by virtue of condition (A_1).

(b) $31\leqslant s\leqslant 35$. With process M_9^s, one finds that Lemma 5.7(I) holds with $u=s-10$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(c) $s=36$. With process N_8^s, one finds that Lemma 5.7(I') holds with $u=s-9$, by virtue of condition (A_1).

(d) $s\geqslant 37$. One finds that process D^s applies.

16. Permissible exponents for fourteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s\geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3\leqslant s\leqslant 16$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l}=0$ $(l=1,2)$ for $3\leqslant s\leqslant 16$. When $j=7$ or 8 and $17\leqslant s\leqslant 20$, meanwhile, we must resort to Theorem 3.4(II)(1). Here we note that condition (iii) is satisfied, and thus the estimate (4.1) holds for $j=7,8$ with $\chi_{j,2}=\frac{1}{3}$ and $\tau_{j,2}=1$. Next, when $j=9$ and $s=21$, we may apply case (β) of Theorem 3.4(Ib) in combination with the condition (iii) to deduce that one may take $\tau_{9,2}=0$.

In order to discuss permissible exponents for $22\leqslant s\leqslant 30$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j=9$ and $l=2$, one has $J=2$, and hence one may take

$$\delta_4=0.0018248,\quad \delta_5=0.0107720,\quad \delta_8=0.1314920,\quad \delta_{10}=0.2835673,$$

whence we may take

$$\tau_{9,l}=\tfrac{1}{9}(\delta_4+\delta_5)<0.00140\quad\text{whenever}\quad \phi_1\leqslant 0.070116\quad (l=2).$$

Also from §23, we find that when $j=9,10,11$ and $12\leqslant j+l\leqslant 13$, one has $J=1$, and hence one may take

$$\delta_4=0.0156211,\quad \delta_5=0.0633584,\quad \delta_6=0.1434849,$$

$$\delta_8=0.3829073,\quad \delta_{10}=0.7106189,\quad \delta_{11}=0.9079284.$$

Thus we deduce that one may take

$$\tau_{9,l}=\tfrac{1}{9}(\delta_4+\delta_5)<0.00878\quad\text{whenever}\quad \phi_1\leqslant 0.068568\quad (l=2,3),$$

that one may take

$$\tau_{10,l}=\tfrac{1}{5}\delta_5<0.01268\quad\text{whenever}\quad \phi_1\leqslant 0.068568\quad (l=2,3),$$

and that one may take

$$\tau_{11,l}=\tfrac{1}{11}\delta_{11}<0.08254\quad (l=2).$$

Table of permissible exponents for $\mathbf{k = 14}$

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0113494	0.00567466			
4	$B_{1,5}^{4,2}$	4.0856057	0.02484611			
5	$A_1^{5,2}$	5.2216967	0.03476683			
6	$B_{2,7}^{6,2}$	6.4118310	0.03979118	0.0160	0.0557	
7	$B_{2,7}^{7,2}$	7.6492691	0.04248943	0.0303	0.0728	
8	$A_2^{8,2}$	8.9350975	0.04500717	0.0383	0.0834	
9	$A_3^{9,2}$	10.2671180	0.04699577	0.0277	0.1175	1.1041
10	$A_3^{10,2}$	11.6442024	0.04876376	0.0373	0.1317	1.2919
11	$B_{4,10}^{11,2}$	13.0662900	0.05051435	0.0268	0.1686	1.4194
12	$A_4^{12,2}$	14.5316795	0.05209364	0.0370	0.1855	1.4979
13	$B_{5,11}^{13,2}$	16.0399382	0.05367993	0.0268	0.2251	1.5676
14	$A_5^{14,2}$	17.5892784	0.05515430	0.0373	0.2442	1.6228
15	$A_6^{15,2}$	19.1785686	0.05660418	0.0273	0.2865	1.6759
16	$A_6^{16,2}$	20.8058286	0.05796461	0.0383	0.3079	1.7234
17	$A_7^{17,2}$	22.4692736	0.05926701	0.0287	0.3527	1.7725
18	$A_7^{18,2}$	24.1667258	0.06048642	0.0398	0.3756	1.8209
19	$A_8^{19,2}$	25.8960050	0.06162954	0.0306	0.4230	1.8711
20	$A_8^{20,2}$	27.6547883	0.06268866	0.0409	0.4458	1.9192
21	$A_9^{21,2}$	29.4407200	0.06366289	0.0276	0.4884	1.9661
22	$A_9^{22,2}$	31.2515093	0.06455699	0.0367	0.5102	2.0125
23	$B_{9,18}^{23,3}$	33.0849211	0.06537338	0.0449	0.5306	2.0596
24	$A_{10}^{24,2}$	34.9387022	0.06610731	0.0337	0.5776	2.1056
25	$A_{10}^{25,2}$	36.8107871	0.06676863	0.0408	0.5967	2.1481
26	$B_{10,20}^{26,3}$	38.6992193	0.06736052	0.0469	0.6137	2.1932
27	$A_{11}^{27,2}$	40.6021475	0.06788535	0.0357	0.6616	2.2350
28	$A_{11}^{28,2}$	42.5178927	0.06835016	0.0410	0.6774	2.2719
29	$A_{11}^{29,2}$	44.4449274	0.06876037	0.0458	0.6916	2.3080
30	$A_{11}^{30,2}$	46.3818646	0.06912079	0.0500	0.7046	2.3493
31	N_{11}^{31}	48.3274553	0.06943614	0.0536	0.7161	2.3867
32	N_{11}^{32}	50.2805449	0.06970829	0.0536	0.7206	2.4066
33	N_{11}^{33}	52.2401670	0.06994609	0.0536	0.7245	2.4242
34	N_{10}^{34}	54.2054937	0.07015541	0.0619	0.6767	2.4527
35	N_{10}^{35}	56.1756866	0.07033184	0.0619	0.6793	2.4637
36	N_{10}^{36}	58.1501035	0.07048574	0.0619	0.6815	2.4733
37	N_{10}^{37}	60.1281620	0.07061847	0.0619	0.6834	2.4817

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
38	M_{10}^{38}	62.1093556	0.07073278	0.0619	0.6851	2.4889
39	N_9^{39}	64.0933213	0.07083658	0.0667	0.6264	2.5149
40	N_9^{40}	66.0795485	0.07091753	0.0667	0.6274	2.5175
$s \geqslant 41$	D^s					

$$\sigma(14) = 0.006895 \quad (s = 34,\ t = 10,\ w = 19),$$

$$\tau(14) = 0.008770 \quad (s = 13), \quad G(14) \leqslant 92 \quad (v = 40).$$

As in the previous cases, our computations for $s \geqslant 31$ depend on first obtaining preliminary estimates by applying the process M_9^s for $31 \leqslant s \leqslant 40$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s \geqslant 41$. In this way we obtain the preliminary permissible exponents

$$\lambda_{31} = 48.328, \quad \lambda_{32} = 50.281, \quad \lambda_{33} = 52.241, \quad \lambda_{34} = 54.206, \quad \lambda_{35} = 56.177,$$

$$\lambda_{36} = 58.151, \quad \lambda_{37} = 60.129, \quad \lambda_{38} = 62.110, \quad \lambda_{39} = 64.094, \quad \lambda_{40} = 66.080,$$

and by virtue of the preliminary exponent

$$\sigma(14) = 0.00689 \quad (s = 34,\ t = 10,\ w = 19),$$

we have also

$$\lambda_s = \max\{2s - 14,\ 66.080 + 2(s - 40)(1 - 0.00689)\}$$

for $s > 40$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $31 \leqslant s \leqslant 33$. With process N_{11}^s, one finds that Lemma 5.7(I') holds with $u = s - 12$, by virtue of condition (A_1).

(b) $34 \leqslant s \leqslant 37$. With process N_{10}^s, one finds that Lemma 5.7(I') holds with $u = s - 11$, by virtue of condition (A_1).

(c) $s = 38$. With process M_{10}^s, one finds that Lemma 5.7(I) holds with $u = s - 11$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(d) $s = 39, 40$. With process N_9^s, one finds that Lemma 5.7(I') holds with $u = s - 10$, by virtue of condition (A_1).

(e) $s \geqslant 41$. One finds that process D^s applies.

17. Permissible exponents for fifteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s\geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3\leqslant s\leqslant 17$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l}=0$ $(l=1,2)$ for $3\leqslant s\leqslant 17$. When $j=7,8,9$ and $18\leqslant s\leqslant 22$, meanwhile, we must resort to Theorem 3.4(II)(1). Here we note that condition (iii) is satisfied, and thus the estimate (4.1) holds for $j=7,8,9$ with $\chi_{j,2}=\frac{1}{3}$ and $\tau_{j,2}=1$. Next, when $j=10$ and $s=23$, we may apply case (β) of Theorem 3.4(Ib) in combination with the condition (iii) to deduce that one may take $\tau_{10,2}=0$.

In order to discuss permissible exponents for $24\leqslant s\leqslant 33$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j=10$ and $l=2$, one has $J=2$, and hence one may take

$$\delta_5=0.0088563 \quad \text{and} \quad \delta_{10}=0.2565541,$$

whence we may take

$$\tau_{10,l}=\tfrac{1}{5}\delta_5<0.00178 \quad \text{whenever} \quad \phi_1\leqslant 0.065621 \quad (l=2).$$

Further, when $j=11,12$ and $13\leqslant j+l\leqslant 14$, one has $J=1$, and hence one may take

$$\delta_5=0.0536213,\quad \delta_6=0.1264298,\quad \delta_{12}=1.0354619.$$

Thus we deduce that one may take

$$\tau_{11,l}=\tfrac{1}{11}(\delta_5+\delta_6)<0.01637 \quad \text{whenever} \quad \phi_1\leqslant 0.063360 \quad (l=2,3),$$

and that one may take

$$\tau_{12,l}=\tfrac{1}{12}\delta_{12}<0.08629 \quad (l=2).$$

As in the previous cases, our computations for $s\geqslant 34$ depend on first obtaining preliminary estimates by applying the process M_{10}^s for $34\leqslant s\leqslant 44$ (noting (5.26) and checking (D_1) or (D_2)), and D^s for $s\geqslant 45$. In this way we obtain the preliminary permissible exponents

$$\lambda_{34}=53.323,\quad \lambda_{35}=55.280,\quad \lambda_{36}=57.243,\quad \lambda_{37}=59.210,$$

$$\lambda_{38}=61.182,\quad \lambda_{39}=63.157,\quad \lambda_{40}=65.136,\quad \lambda_{41}=67.118,$$

$$\lambda_{42}=69.102,\quad \lambda_{43}=71.088,\quad \lambda_{44}=73.076,$$

and by virtue of the preliminary exponent

$$\sigma(15) = 0.00633 \quad (s = 37,\ t = 11,\ w = 21),$$

we have also

$$\lambda_s = \max\{2s - 15,\ 73.076 + 2(s - 44)(1 - 0.00633)\}$$

for $s > 44$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 34, 35$. With process N^s_{12}, one finds that Lemma 5.7(I') holds with $u = s - 13$, by virtue of condition (A_1).

(b) $36 \leqslant s \leqslant 40$. With process N^s_{11}, one finds that Lemma 5.7(I') holds with $u = s - 12$, by virtue of condition (A_1).

(c) $s = 41$. With process M^s_{11}, one finds that Lemma 5.7(I) holds with $u = s - 12$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(d) $s = 42, 43$. With process N^s_{10}, one finds that Lemma 5.7(I') holds with $u = s - 11$, by virtue of condition (A_1).

(e) $s \geqslant 44$. One finds that process D^s applies.

Table of permissible exponents for k = 15

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ^*_s
3	$B^{3,2}_{1,4}$	3.0092359	0.00461792			
4	$B^{4,2}_{1,6}$	4.0742295	0.02173146			
5	$A^{5,2}_{1}$	5.1987873	0.03172823			
6	$B^{6,2}_{2,8}$	6.3738989	0.03647237	0.0116	0.0481	
7	$B^{7,2}_{2,7}$	7.5936427	0.03905794	0.0269	0.0659	
8	$A^{8,2}_{2}$	8.8571841	0.04113748	0.0342	0.0753	
9	$B^{9,2}_{3,9}$	10.1649995	0.04309441	0.0236	0.1058	1.0461
10	$A^{10,2}_{3}$	11.5141981	0.04456907	0.0327	0.1189	1.2336
11	$B^{11,2}_{4,10}$	12.9059619	0.04616698	0.0222	0.1517	1.3792
12	$A^{12,2}_{4}$	14.3381411	0.04752336	0.0321	0.1674	1.4522
13	$B^{13,2}_{5,11}$	15.8111209	0.04895330	0.0223	0.2032	1.5241
14	$A^{14,2}_{5}$	17.3230870	0.05024754	0.0321	0.2205	1.5749
15	$A^{15,2}_{6}$	18.8736139	0.05156238	0.0229	0.2592	1.6271
16	$A^{16,2}_{6}$	20.4609219	0.05278515	0.0331	0.2784	1.6717
17	$A^{17,2}_{7}$	22.0838491	0.05398415	0.0239	0.3190	1.7169
18	$A^{18,2}_{7}$	23.7405373	0.05510909	0.0340	0.3394	1.7623
19	$A^{19,2}_{8}$	25.4293639	0.05618735	0.0257	0.3830	1.8090
20	$A^{20,2}_{8}$	27.1483206	0.05719334	0.0356	0.4044	1.8549
21	$A^{21,2}_{9}$	28.8955107	0.05813950	0.0271	0.4492	1.9020
22	$A^{22,2}_{9}$	30.6688816	0.05901573	0.0367	0.4710	1.9480

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
23	$A_{10}^{23,2}$	32.4664265	0.05982581	0.0239	0.5104	1.9912
24	$A_{10}^{24,2}$	34.2861492	0.06056957	0.0320	0.5305	2.0351
25	$A_{10}^{25,2}$	36.1261310	0.06125062	0.0397	0.5499	2.0782
26	$A_{11}^{26,2}$	37.9844896	0.06186873	0.0290	0.5938	2.1226
27	$A_{11}^{27,2}$	39.8594364	0.06242704	0.0358	0.6119	2.1634
28	$B_{11,22}^{28,3}$	41.7493055	0.06293025	0.0417	0.6284	2.2023
29	$A_{12}^{29,2}$	43.6525286	0.06338099	0.0310	0.6730	2.2453
30	$A_{12}^{30,2}$	45.5676489	0.06378269	0.0359	0.6878	2.2808
31	$A_{12}^{31,2}$	47.4933457	0.06414041	0.0405	0.7015	2.3144
32	$A_{12}^{32,2}$	49.4284091	0.06445755	0.0444	0.7139	2.3488
33	$A_{12}^{33,2}$	51.3717431	0.06473789	0.0479	0.7251	2.3859
34	N_{12}^{34}	53.3223547	0.06498462	0.0500	0.7335	2.4158
35	N_{12}^{35}	55.2793490	0.06520081	0.0500	0.7376	2.4338
36	N_{11}^{36}	57.2419678	0.06539241	0.0578	0.6939	2.4654
37	N_{11}^{37}	59.2094705	0.06555771	0.0578	0.6966	2.4771
38	N_{11}^{38}	61.1812515	0.06570293	0.0578	0.6991	2.4875
39	N_{11}^{39}	63.1567645	0.06582965	0.0578	0.7012	2.4965
40	N_{11}^{40}	65.1355287	0.06594009	0.0578	0.7031	2.5045
41	M_{11}^{41}	67.1171222	0.06603623	0.0578	0.7047	2.5115
42	N_{10}^{42}	69.1012148	0.06612247	0.0623	0.6500	2.5383
43	N_{10}^{43}	71.0874163	0.06619342	0.0623	0.6511	2.5409
$s \geqslant 44$	D^s					

$$\sigma(15) = 0.006338 \quad (s = 37,\ t = 11,\ w = 21),$$

$$\tau(15) = 0.008114 \quad (s = 14), \quad G(15) \leqslant 100 \quad (v = 43).$$

18. Permissible exponents for sixteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3 \leqslant s \leqslant 18$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ for $3 \leqslant s \leqslant 18$. When $j = 8, 9, 10$ and $19 \leqslant s \leqslant 24$, meanwhile, we must resort to Theorem 3.4(II)(1). Here we note that condition (iii) is satisfied, and thus the estimate (4.1) holds for $j = 8, 9, 10$ with $\chi_{j,2} = \frac{1}{3}$ and $\tau_{j,2} = 1$.

In order to discuss permissible exponents for $25 \leqslant s \leqslant 36$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 11$ and $l = 1$ or 2, one has $J = 2$, and hence one may take

$$\delta_5 = 0.0073658, \quad \delta_6 = 0.0241263 \quad \text{and} \quad \delta_{12} = 0.4026992,$$

whence we may take

$$\tau_{11,l} = \tfrac{1}{11}(\delta_5 + \delta_6) < 0.00287 \quad \text{whenever} \quad \phi_1 \leqslant 0.061146 \quad (l = 1, 2).$$

Further, when $j = 11, 12, 13$ and $14 \leqslant j + l \leqslant 15$, one has $J = 1$, and hence one may take

$$\delta_5 = 0.0460456, \quad \delta_6 = 0.1127790, \quad \delta_{12} = 0.9585035, \quad \delta_{13} = 1.1659645.$$

Thus we deduce that one may take

$$\tau_{11,l} = \tfrac{1}{11}(\delta_5 + \delta_6) < 0.01444 \quad \text{whenever} \quad \phi_1 \leqslant 0.059762 \quad (l = 3),$$

that one may take

$$\tau_{12,l} = \tfrac{1}{6}\delta_6 < 0.01880 \quad \text{whenever} \quad \phi_1 \leqslant 0.059762 \quad (l = 2),$$

and that one may take

$$\tau_{13,l} = \tfrac{1}{13}\delta_{13} < 0.08969 \quad (l = 2).$$

As in the previous cases, our computations for $s \geqslant 37$ depend on first obtaining preliminary estimates by applying the process M_{10}^s for $37 \leqslant s \leqslant 47$ (noting (5.27) and checking (D_1) or (D_2)), and D^s for $s \geqslant 48$. In this way we obtain the preliminary permissible exponents

$$\lambda_{37} = 58.320, \quad \lambda_{38} = 60.280, \quad \lambda_{39} = 62.245, \quad \lambda_{40} = 64.215,$$

$$\lambda_{41} = 66.188, \quad \lambda_{42} = 68.164, \quad \lambda_{43} = 70.143, \quad \lambda_{44} = 72.125,$$

$$\lambda_{45} = 74.110, \quad \lambda_{46} = 76.096, \quad \lambda_{47} = 78.084,$$

and by virtue of the preliminary exponent

$$\sigma(16) = 0.00586 \quad (s = 41,\ t = 11,\ w = 23),$$

we have also

$$\lambda_s = \max\{2s - 16,\ 78.084 + 2(s - 47)(1 - 0.00586)\}$$

for $s > 47$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 37, 38$. With process N_{13}^s, one finds that Lemma 5.7(I') holds with $u = s - 14$, by virtue of condition (A_1).

(b) $39 \leqslant s \leqslant 43$. With process N_{12}^s, one finds that Lemma 5.7(I') holds with $u = s - 13$, by virtue of condition (A_1).

(c) $s = 44$. With process M_{12}^s, one finds that Lemma 5.7(I) holds with $u = s-13$, by virtue of conditions (D_1), (C_1), (C_2), (C_3).

(d) $45 \leqslant s \leqslant 47$. With process N_{11}^s, one finds that Lemma 5.7(I') holds with $u = s - 12$, by virtue of condition (A_1).

(e) $s \geqslant 48$. One finds that process D^s applies.

Table of permissible exponents for k = 16

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0074816	0.00374079			
4	$B_{1,6}^{4,2}$	4.0638321	0.01883046			
5	$A_1^{5,2}$	5.1787385	0.02919246			
6	$B_{2,8}^{6,2}$	6.3412246	0.03370200	0.0083	0.0420	
7	$B_{2,8}^{7,2}$	7.5457229	0.03613827	0.0241	0.0602	
8	$A_2^{8,2}$	8.7902441	0.03788514	0.0308	0.0687	
9	$B_{3,9}^{9,2}$	10.0769155	0.03976160	0.0200	0.0957	0.9893
10	$A_3^{10,2}$	11.4019639	0.04102548	0.0289	0.1082	1.1825
11	$B_{4,10}^{11,2}$	12.7671799	0.04247669	0.0185	0.1375	1.3449
12	$A_4^{12,2}$	14.1702162	0.04365257	0.0280	0.1522	1.4130
13	$B_{5,12}^{13,2}$	15.6119303	0.04493631	0.0182	0.1842	1.4861
14	$A_5^{14,2}$	17.0906280	0.04608149	0.0276	0.2003	1.5333
15	$B_{6,13}^{15,2}$	18.6062852	0.04726736	0.0188	0.2354	1.5841
16	$A_6^{16,2}$	20.1573791	0.04836825	0.0285	0.2531	1.6259
17	$B_{7,14}^{17,2}$	21.7431712	0.04946474	0.0198	0.2902	1.6685
18	$A_7^{18,2}$	23.3621148	0.05049786	0.0293	0.3090	1.7107
19	$A_8^{19,2}$	25.0129891	0.05150184	0.0211	0.3485	1.7539
20	$A_8^{20,2}$	26.6941169	0.05244686	0.0306	0.3682	1.7972
21	$A_9^{21,2}$	28.4039654	0.05334847	0.0229	0.4098	1.8413
22	$A_9^{22,2}$	30.1407641	0.05419218	0.0322	0.4304	1.8854
23	$A_{10}^{23,2}$	31.9028113	0.05498480	0.0243	0.4729	1.9297
24	$A_{10}^{24,2}$	33.6883143	0.05572055	0.0329	0.4931	1.9734
25	$B_{11,19}^{25,2}$	35.4955330	0.05640278	0.0205	0.5297	2.0143
26	$A_{11}^{26,2}$	37.3227332	0.05703073	0.0279	0.5485	2.0548
27	$A_{11}^{27,2}$	39.1682523	0.05760740	0.0351	0.5668	2.0961
28	$B_{11,21}^{28,3}$	41.0304795	0.05813390	0.0408	0.5828	2.1367
29	$A_{12}^{29,2}$	42.9078717	0.05861191	0.0313	0.6252	2.1765
30	$A_{12}^{30,2}$	44.7989875	0.05904508	0.0369	0.6410	2.2140
31	$B_{12,24}^{31,3}$	46.7024729	0.05943588	0.0414	0.6547	2.2506
32	$A_{13}^{32,2}$	48.6170673	0.05978708	0.0316	0.6971	2.2884
33	$A_{13}^{33,2}$	50.5416092	0.06010180	0.0358	0.7101	2.3210

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
34	$A_{13}^{34,2}$	52.4750359	0.06038318	0.0396	0.7220	2.3515
35	$A_{13}^{35,2}$	54.4163771	0.06063404	0.0429	0.7329	2.3839
36	$A_{13}^{36,2}$	56.3647518	0.06085714	0.0459	0.7428	2.4179
37	N_{13}^{37}	58.3193587	0.06105480	0.0469	0.7488	2.4413
38	N_{13}^{38}	60.2794807	0.06122977	0.0469	0.7526	2.4577
39	N_{12}^{39}	62.2444892	0.06138528	0.0543	0.7115	2.4878
40	N_{12}^{40}	64.2137908	0.06152144	0.0543	0.7141	2.4987
41	N_{12}^{41}	66.1868812	0.06164181	0.0543	0.7164	2.5084
42	N_{12}^{42}	68.1633061	0.06174778	0.0543	0.7185	2.5169
43	N_{12}^{43}	70.1426626	0.06184097	0.0543	0.7203	2.5246
44	M_{12}^{44}	72.1245941	0.06192285	0.0543	0.7219	2.5313
45	N_{11}^{45}	74.1088058	0.06199601	0.0584	0.6708	2.5587
46	N_{11}^{46}	76.0949856	0.06205826	0.0584	0.6718	2.5613
47	N_{11}^{47}	78.0829008	0.06211345	0.0584	0.6727	2.5636
$s\geqslant 48$	D^s					

$$\sigma(16) = 0.005864 \quad (s = 41,\, t = 11,\, w = 23),$$

$$\tau(16) = 0.007549 \quad (s = 15), \quad G(16)\leqslant 109 \quad (v = 47).$$

19. Permissible exponents for seventeenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s\geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3\leqslant s\leqslant 19$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ ($l = 1, 2$) for $3\leqslant s\leqslant 19$. When $8\leqslant j\leqslant 11$ and $20\leqslant s\leqslant 27$, meanwhile, we must resort to Theorem 3.4(II)(1) and (2). Here we note that condition (iii) is satisfied for $20\leqslant s\leqslant 26$, and that (II)(2) applies for $s = 27$, and thus the estimate (4.1) holds for $8\leqslant j\leqslant 11$ and $l = 2, 3$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Note here that when $s = 27$, the relevant value of σ is so small that the condition $\phi_1 + \cdots + \phi_j\geqslant \frac{1}{3}(1-\sigma)^{-1}$ is satisfied transparently.

In order to discuss permissible exponents for $28\leqslant s\leqslant 39$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 12$ and $l = 2$, one has $J = 2$, and hence one may take

$$\delta_6 = 0.0205621 \quad \text{and} \quad \delta_{12} = 0.3706630,$$

whence we may take

$$\tau_{12,l} = \tfrac{1}{6}\delta_6 < 0.00343 \quad \text{whenever} \quad \phi_1\leqslant 0.057704 \quad (l = 2).$$

Further, when $j = 12, 13, 14$ and $15 \leqslant j + l \leqslant 16$, one has $J = 1$, and hence one may take

$$\delta_6 = 0.1004200, \quad \delta_7 = 0.1840767, \quad \delta_{12} = 0.8905163,$$
$$\delta_{13} = 1.0847742, \quad \delta_{14} = 1.2966345.$$

Thus we deduce that one may take

$$\tau_{12,l} = \tfrac{1}{6}\delta_6 < 0.01674 \quad \text{whenever} \quad \phi_1 \leqslant 0.056530 \quad (l = 3),$$

that one may take

$$\tau_{13,l} = \tfrac{1}{13}(\delta_6 + \delta_7) < 0.02189 \quad \text{whenever} \quad \phi_1 \leqslant 0.055777 \quad (l = 2),$$

and otherwise,

$$\tau_{13,l} = \tfrac{1}{13}\delta_{13} < 0.08345 \quad (l = 2, 3),$$

and that one may take

$$\tau_{14,l} = \tfrac{1}{14}\delta_{14} < 0.09262 \quad (l = 2).$$

As in the previous cases, our computations for $s \geqslant 40$ depend on first obtaining preliminary estimates by applying the process M_{11}^s for $40 \leqslant s \leqslant 51$ (noting (5.27) and checking (D_1) or (D_2)), and D^s for $s \geqslant 52$. In this way we obtain the preliminary permissible exponents

$$\lambda_{40} = 63.318, \quad \lambda_{41} = 65.281, \quad \lambda_{42} = 67.248, \quad \lambda_{43} = 69.219,$$
$$\lambda_{44} = 71.193, \quad \lambda_{45} = 73.170, \quad \lambda_{46} = 75.150, \quad \lambda_{47} = 77.133,$$
$$\lambda_{48} = 79.117, \quad \lambda_{49} = 81.103, \quad \lambda_{50} = 83.091, \quad \lambda_{51} = 85.080,$$

and by virtue of the preliminary exponent

$$\sigma(17) = 0.00545 \quad (s = 44,\ t = 12,\ w = 24),$$

we have also

$$\lambda_s = \max\{2s - 17,\ 85.080 + 2(s - 51)(1 - 0.00545)\}$$

for $s > 51$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 40, 41$. With process N_{14}^s, one finds that Lemma 5.7(I') holds with $u = s - 15$, by virtue of condition (A_1).

(b) $42 \leqslant s \leqslant 46$. With process N_{13}^s, one finds that Lemma 5.7(I') holds with $u = s - 14$, by virtue of condition (A_1).

(c) $47 \leqslant s \leqslant 50$. With process N_{12}^s, one finds that Lemma 5.7(I') holds with $u = s - 13$, by virtue of condition (A_1).

(e) $s \geqslant 51$. One finds that process D^s applies.

Table of permissible exponents for k = 17

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0061404	0.00307019			
4	$B_{1,6}^{4,2}$	4.0554335	0.01646473			
5	$A_1^{5,2}$	5.1619828	0.02701166			
6	$B_{2,8}^{6,2}$	6.3136071	0.03134018	0.0056	0.0369	
7	$B_{2,8}^{7,2}$	7.5040992	0.03349964	0.0210	0.0545	
8	$A_2^{8,2}$	8.7323022	0.03513032	0.0278	0.0630	
9	$B_{3,10}^{9,2}$	10.0004288	0.03689292	0.0168	0.0869	0.9360
10	$A_3^{10,2}$	11.3043577	0.03799316	0.0257	0.0991	1.1377
11	$A_3^{11,2}$	12.6459140	0.03927903	0.0330	0.1097	1.3053
12	$A_4^{12,2}$	14.0233197	0.04034662	0.0245	0.1392	1.3790
13	$A_4^{13,2}$	15.4369512	0.04145984	0.0321	0.1516	1.4456
14	$A_5^{14,2}$	16.8860867	0.04251950	0.0239	0.1831	1.4968
15	$A_5^{15,2}$	18.3702735	0.04356583	0.0325	0.1980	1.5438
16	$A_6^{16,2}$	19.8886761	0.04457565	0.0244	0.2314	1.5855
17	$A_6^{17,2}$	21.4405373	0.04556572	0.0332	0.2477	1.6256
18	$A_7^{18,2}$	23.0247863	0.04651864	0.0251	0.2827	1.6648
19	$A_7^{19,2}$	24.6403737	0.04744334	0.0339	0.3001	1.7059
20	$A_8^{20,2}$	26.2860071	0.04832721	0.0259	0.3366	1.7458
21	$B_{8,17}^{21,3}$	27.9604002	0.04917555	0.0350	0.3554	1.7878
22	$A_9^{22,2}$	29.6620874	0.04997915	0.0279	0.3945	1.8289
23	$B_{9,18}^{23,3}$	31.3896134	0.05074142	0.0362	0.4131	1.8710
24	$A_{10}^{24,2}$	33.1414245	0.05145731	0.0291	0.4534	1.9126
25	$B_{10,19}^{25,3}$	34.9159903	0.05212921	0.0368	0.4717	1.9544
26	$A_{11}^{26,2}$	36.7117441	0.05275480	0.0295	0.5126	1.9959
27	$B_{11,20}^{27,3}$	38.5271601	0.05333611	0.0367	0.5307	2.0372
28	$A_{12}^{28,2}$	40.3607211	0.05387253	0.0245	0.5647	2.0725
29	$A_{12}^{29,2}$	42.2109803	0.05436691	0.0310	0.5818	2.1119
30	$B_{12,23}^{30,3}$	44.0765377	0.05482022	0.0367	0.5976	2.1499
31	$A_{13}^{31,2}$	45.9560566	0.05523415	0.0275	0.6371	2.1879
32	$A_{13}^{32,2}$	47.8482757	0.05561097	0.0328	0.6521	2.2240
33	$B_{13,25}^{33,3}$	49.7520144	0.05595308	0.0379	0.6667	2.2594
34	$B_{13,25}^{34,3}$	51.6661665	0.05626250	0.0409	0.6775	2.2946
35	$A_{14}^{35,2}$	53.5897087	0.05654168	0.0318	0.7177	2.3262
36	$A_{14}^{36,2}$	55.5216990	0.05679304	0.0354	0.7292	2.3562
37	$A_{14}^{37,2}$	57.4612719	0.05701880	0.0386	0.7398	2.3843
38	$A_{14}^{38,2}$	59.4076368	0.05722114	0.0414	0.7493	2.4146
39	$A_{14}^{39,2}$	61.3600741	0.05740216	0.0439	0.7581	2.4458

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
40	N_{14}^{40}	63.3179278	0.05756359	0.0442	0.7624	2.4638
41	N_{14}^{41}	65.2806096	0.05770757	0.0442	0.7658	2.4789
42	N_{13}^{42}	67.2475940	0.05783611	0.0511	0.7271	2.5076
43	N_{13}^{43}	69.2183942	0.05794990	0.0511	0.7296	2.5178
44	N_{13}^{44}	71.1925852	0.05805112	0.0511	0.7318	2.5269
45	N_{13}^{45}	73.1697840	0.05814094	0.0511	0.7337	2.5350
46	N_{13}^{46}	75.1496484	0.05822056	0.0511	0.7355	2.5423
47	N_{12}^{47}	77.1318848	0.05829175	0.0550	0.6878	2.5738
48	N_{12}^{48}	79.1162028	0.05835376	0.0550	0.6890	2.5767
49	N_{12}^{49}	81.1023673	0.05840894	0.0550	0.6901	2.5793
50	N_{12}^{50}	83.0901643	0.05845772	0.0550	0.6910	2.5816
$s \geqslant 51$	D^s					

$$\sigma(17) = 0.005454 \quad (s = 44,\ t = 12,\ w = 24),$$

$$\tau(17) = 0.007060 \quad (s = 16), \quad G(17) \leqslant 117 \quad (v = 50).$$

20. Permissible exponents for eighteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3 \leqslant s \leqslant 21$, the condition ($\delta$) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ ($l = 1, 2, 3$) for $3 \leqslant s \leqslant 21$. When $9 \leqslant j \leqslant 12$ and $22 \leqslant s \leqslant 29$, meanwhile, we must resort to Theorem 3.4(II)(1) and (2). Here we note that condition (iii) is satisfied for $22 \leqslant s \leqslant 27$, and that (II)(2) applies for $s = 28, 29$, and thus the estimate (4.1) holds for $9 \leqslant j \leqslant 12$ and $l = 2, 3$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Note here that when $s = 28, 29$, the relevant value of σ is so small that the condition $\phi_1 + \cdots + \phi_j \geqslant \frac{1}{3}(1 - \sigma)^{-1}$ is satisfied transparently.

In order to discuss permissible exponents for $30 \leqslant s \leqslant 41$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 13$ and $l = 2$, one has $J = 2$, and hence one may take

$$\delta_6 = 0.0176775, \quad \delta_7 = 0.0427686 \quad \text{and} \quad \delta_{14} = 0.5236334,$$

whence we may take

$$\tau_{13,l} = \tfrac{1}{13}(\delta_6 + \delta_7) < 0.00465 \quad \text{whenever} \quad \phi_1 \leqslant 0.054235 \quad (l = 2).$$

Further, when $j = 13, 14, 15$ and $16 \leqslant j + l \leqslant 17$, one has $J = 1$, and hence one may take

$$\delta_6 = 0.0901459, \quad \delta_7 = 0.1679365, \quad \delta_{14} = 1.2130601, \quad \delta_{15} = 1.4286845.$$

Thus we deduce that one may take

$$\tau_{13,l} = \tfrac{1}{13}(\delta_6 + \delta_7) < 0.01986 \quad \text{whenever} \quad \phi_1 \leqslant 0.052973 \quad (l = 3),$$

that one may take

$$\tau_{14,l} = \tfrac{1}{7}\delta_7 < 0.02400 \quad \text{whenever} \quad \phi_1 \leqslant 0.052973 \quad (l = 2, 3),$$

and otherwise,

$$\tau_{14,l} = \tfrac{1}{14}\delta_{14} < 0.08665 \quad (l = 2, 3),$$

and that one may take

$$\tau_{15,l} = \tfrac{1}{15}\delta_{15} < 0.09525 \quad (l = 2).$$

As in the previous cases, our computations for $s \geqslant 42$ depend on first obtaining preliminary estimates by applying the process M_{12}^s for $42 \leqslant s \leqslant 54$ (noting (5.27) and checking (D_1) or (D_2)), and D^s for $s \geqslant 55$. In this way we obtain the preliminary permissible exponents

$$\lambda_{42} = 66.358, \quad \lambda_{43} = 68.318, \quad \lambda_{44} = 70.283, \quad \lambda_{45} = 72.252, \quad \lambda_{46} = 74.224,$$

$$\lambda_{47} = 76.199, \quad \lambda_{48} = 78.177, \quad \lambda_{49} = 80.157, \quad \lambda_{50} = 82.140,$$
$$\lambda_{51} = 84.124, \quad \lambda_{52} = 86.110, \quad \lambda_{53} = 88.098, \quad \lambda_{54} = 90.087,$$

and by virtue of the preliminary exponent

$$\sigma(18) = 0.00509 \quad (s = 47,\ t = 13,\ w = 26),$$

we have also

$$\lambda_s = \max\{2s - 18,\ 90.087 + 2(s - 54)(1 - 0.00509)\}$$

for $s > 54$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $42 \leqslant s \leqslant 44$. With process N_{15}^s, one finds that Lemma 5.7(I') holds with $u = s - 16$, by virtue of condition (A_1).

(b) $45 \leqslant s \leqslant 49$. With process N_{14}^s, one finds that Lemma 5.7(I') holds with $u = s - 15$, by virtue of condition (A_1).

(c) $50 \leqslant s \leqslant 54$. With process N_{13}^s, one finds that Lemma 5.7(I') holds with $u = s - 14$, by virtue of condition (A_1).

(e) $s \geqslant 55$. One finds that process D^s applies.

Table of permissible exponents for k = 18

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0050870	0.00254346			
4	$B_{1,6}^{4,2}$	4.0484489	0.01447855			
5	$B_{1,6}^{5,2}$	5.1469671	0.02493153			
6	$B_{2,8}^{6,2}$	6.2893077	0.02933022	0.0034	0.0328	
7	$B_{2,8}^{7,2}$	7.4676627	0.03123178	0.0185	0.0497	
8	$A_{2}^{8,2}$	8.6816032	0.03275099	0.0254	0.0581	
9	$B_{3,10}^{9,2}$	9.9332381	0.03438389	0.0135	0.0787	0.8757
10	$A_{3}^{10,2}$	11.2185989	0.03537489	0.0230	0.0913	1.0979
11	$A_{3}^{11,2}$	12.5388988	0.03647482	0.0295	0.1006	1.2547
12	$B_{4,11}^{12,2}$	13.8936352	0.03749420	0.0213	0.1278	1.3486
13	$A_{4}^{13,2}$	15.2823305	0.03846045	0.0284	0.1390	1.4100
14	$A_{5}^{14,2}$	16.7049932	0.03943607	0.0206	0.1683	1.4650
15	$A_{5}^{15,2}$	18.1609021	0.04036376	0.0285	0.1816	1.5081
16	$A_{6}^{16,2}$	19.6497248	0.04128885	0.0209	0.2126	1.5495
17	$A_{6}^{17,2}$	21.1706906	0.04218253	0.0290	0.2273	1.5872
18	$A_{7}^{18,2}$	22.7231221	0.04306012	0.0215	0.2599	1.6240
19	$A_{7}^{19,2}$	24.3060925	0.04390870	0.0296	0.2757	1.6626
20	$A_{8}^{20,2}$	25.9186447	0.04473174	0.0222	0.3097	1.7001
21	$B_{8,17}^{21,3}$	27.5596769	0.04552348	0.0306	0.3268	1.7397
22	$A_{9}^{22,2}$	29.2280189	0.04628304	0.0237	0.3628	1.7781
23	$B_{9,18}^{23,3}$	30.9224199	0.04700798	0.0321	0.3807	1.8185
24	$A_{10}^{24,2}$	32.6415685	0.04769656	0.0253	0.4180	1.8574
25	$B_{10,19}^{25,3}$	34.3841221	0.04834828	0.0331	0.4359	1.8979
26	$A_{11}^{26,2}$	36.1487034	0.04896179	0.0263	0.4739	1.9370
27	$B_{11,20}^{27,3}$	37.9339369	0.04953749	0.0337	0.4917	1.9773
28	$A_{12}^{28,2}$	39.7384452	0.05007502	0.0266	0.5304	2.0163
29	$B_{12,22}^{29,3}$	41.5608774	0.05057526	0.0334	0.5475	2.0558
30	$A_{13}^{30,2}$	43.3999089	0.05103871	0.0214	0.5792	2.0894
31	$A_{13}^{31,2}$	45.2542644	0.05146692	0.0274	0.5952	2.1257
32	$B_{13,25}^{32,3}$	47.1227175	0.05186115	0.0330	0.6106	2.1626
33	$B_{13,24}^{33,3}$	49.0040975	0.05222287	0.0371	0.6234	2.1972
34	$A_{14}^{34,2}$	50.8972954	0.05255372	0.0291	0.6620	2.2327
35	$A_{14}^{35,2}$	52.8012695	0.05285563	0.0335	0.6753	2.2662
36	$B_{14,27}^{36,3}$	54.7150443	0.05313037	0.0374	0.6874	2.2979
37	$A_{15}^{37,2}$	56.6377106	0.05337974	0.0283	0.7246	2.3305
38	$A_{15}^{38,2}$	58.5684269	0.05360563	0.0317	0.7356	2.3597
39	$A_{15}^{39,2}$	60.5064172	0.05380985	0.0347	0.7458	2.3872

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
40	$A_{15}^{40,2}$	62.4509681	0.05399414	0.0375	0.7552	2.4132
41	$A_{15}^{41,2}$	64.4014262	0.05416015	0.0399	0.7637	2.4418
42	N_{15}^{42}	66.3571950	0.05430945	0.0417	0.7709	2.4685
43	N_{15}^{43}	68.3177304	0.05444347	0.0417	0.7744	2.4838
44	N_{15}^{44}	70.2825406	0.05456372	0.0417	0.7776	2.4977
45	N_{14}^{45}	72.2511825	0.05467161	0.0483	0.7410	2.5253
46	N_{14}^{46}	74.2232486	0.05476794	0.0483	0.7433	2.5349
47	N_{14}^{47}	76.1983771	0.05485415	0.0483	0.7454	2.5434
48	N_{14}^{48}	78.1762412	0.05493118	0.0483	0.7473	2.5512
49	N_{14}^{49}	80.1565468	0.05499994	0.0483	0.7490	2.5581
50	N_{13}^{50}	82.1390360	0.05506161	0.0519	0.7040	2.5899
51	N_{13}^{51}	84.1234628	0.05511611	0.0519	0.7052	2.5927
52	N_{13}^{52}	86.1096189	0.05516484	0.0519	0.7063	2.5953
53	N_{13}^{53}	88.0973151	0.05520823	0.0519	0.7072	2.5975
54	N_{13}^{54}	90.0863822	0.05524686	0.0519	0.7081	2.5996
$s\geqslant 55$	D^s					

$$\sigma(18) = 0.005095 \quad (s = 47,\, t = 13,\, w = 26),$$

$$\tau(18) = 0.006630 \quad (s = 17), \quad G(18)\leqslant 125 \quad (v = 54).$$

21. Permissible exponents for nineteenth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s\geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3\leqslant s\leqslant 21$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ ($l = 1, 2$) for $3\leqslant s\leqslant 21$. When $9\leqslant j\leqslant 13$ and $22\leqslant s\leqslant 31$, meanwhile, we must resort to Theorem 3.4(II)(1) and (2). Here we note that condition (iii) is satisfied for $22\leqslant s\leqslant 29$, and that (II)(2) applies for $s = 30, 31$, and thus the estimate (4.1) holds for $9\leqslant j\leqslant 13$ and $l = 2, 3$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Note here that when $s = 30, 31$, the relevant value of σ is so small that the condition $\phi_1 + \cdots + \phi_j\geqslant\frac{1}{3}(1-\sigma)^{-1}$ is satisfied transparently.

In order to discuss permissible exponents for $32\leqslant s\leqslant 44$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 14$ and $l = 2$, one has $J = 2$, and hence one may take

$$\delta_7 = 0.0375713, \quad \text{and} \quad \delta_{14} = 0.4889692,$$

whence we may take

$$\tau_{14,l} = \tfrac{1}{7}\delta_7 < 0.00537 \quad \text{whenever} \quad \phi_1\leqslant 0.051509 \quad (l = 2).$$

Further, when $j = 14, 15$ and $17 \leqslant j + l \leqslant 18$, one has $J = 1$, and hence one may take

$$\delta_7 = 0.1541435, \quad \delta_8 = 0.2475913, \quad \delta_{14} = 1.1394461,$$

$$\delta_{15} = 1.3428361, \quad \delta_{16} = 1.5615789.$$

Thus we deduce that one may take

$$\tau_{14,l} = \tfrac{1}{7}\delta_7 < 0.02203 \quad \text{whenever} \quad \phi_1 \leqslant 0.050425 \quad (l = 3),$$

that one may take

$$\tau_{15,l} = \tfrac{1}{15}(\delta_7 + \delta_8) < 0.02679 \quad \text{whenever} \quad \phi_1 \leqslant 0.049834 \quad (l = 2, 3),$$

and otherwise,

$$\tau_{15,l} = \tfrac{1}{15}\delta_{15} < 0.08953 \quad (l = 2, 3).$$

As in the previous cases, our computations for $s \geqslant 45$ depend on first obtaining preliminary estimates by applying the process M_{13}^s for $45 \leqslant s \leqslant 58$ (noting (5.27) and checking (D_1) or (D_2)), and D^s for $s \geqslant 59$. In this way we obtain the preliminary permissible exponents

$$\lambda_{45} = 71.356, \quad \lambda_{46} = 73.319, \quad \lambda_{47} = 75.286, \quad \lambda_{48} = 77.256, \quad \lambda_{49} = 79.229,$$

$$\lambda_{50} = 81.205, \quad \lambda_{51} = 83.183, \quad \lambda_{52} = 85.164, \quad \lambda_{53} = 87.147, \quad \lambda_{54} = 89.131,$$

$$\lambda_{55} = 91.117, \quad \lambda_{56} = 93.105, \quad \lambda_{57} = 95.094, \quad \lambda_{58} = 97.084,$$

and by virtue of the preliminary exponent

$$\sigma(19) = 0.00478 \quad (s = 50,\ t = 13,\ w = 28),$$

we have also

$$\lambda_s = \max\{2s - 19,\ 97.084 + 2(s - 58)(1 - 0.00478)\}$$

for $s > 58$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $45 \leqslant s \leqslant 47$. With process N_{16}^s, one finds that Lemma 5.7(I') holds with $u = s - 17$, by virtue of condition (A_1).

(b) $48 \leqslant s \leqslant 52$. With process N_{15}^s, one finds that Lemma 5.7(I') holds with $u = s - 16$, by virtue of condition (A_1).

(c) $53 \leqslant s \leqslant 58$. With process N_{14}^s, one finds that Lemma 5.7(I') holds with $u = s - 15$, by virtue of condition (A_1).

(e) $s \geqslant 59$. One finds that process D^s applies.

Table of permissible exponents for k = 19

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0042273	0.00211363			
4	$B_{1,6}^{4,2}$	4.0423620	0.01272951			
5	$B_{1,6}^{5,2}$	5.1333114	0.02298073			
6	$A_1^{6,2}$	6.2663273	0.02733191			
7	$B_{2,8}^{7,2}$	7.4343072	0.02929710	0.0164	0.0457	
8	$A_2^{8,2}$	8.6357245	0.03067723	0.0233	0.0540	
9	$A_2^{9,2}$	9.8725415	0.03215755	0.0286	0.0608	
10	$B_{3,10}^{10,2}$	11.1414574	0.03308734	0.0205	0.0843	1.0573
11	$A_3^{11,2}$	12.4430359	0.03404381	0.0266	0.0929	1.2102
12	$B_{4,11}^{12,2}$	13.7775620	0.03500339	0.0186	0.1180	1.3222
13	$A_4^{13,2}$	15.1440547	0.03585180	0.0252	0.1282	1.3792
14	$A_5^{14,2}$	16.5429734	0.03674656	0.0179	0.1555	1.4369
15	$A_5^{15,2}$	17.9734687	0.03757479	0.0250	0.1674	1.4766
16	$B_{6,14}^{16,2}$	19.4355306	0.03842022	0.0178	0.1962	1.5175
17	$A_6^{17,2}$	20.9284296	0.03922960	0.0253	0.2096	1.5529
18	$B_{7,15}^{18,2}$	22.4517574	0.04003558	0.0182	0.2399	1.5885
19	$A_7^{19,2}$	24.0047167	0.04081410	0.0259	0.2544	1.6239
20	$A_8^{20,2}$	25.5866021	0.04157726	0.0189	0.2861	1.6601
21	$A_8^{21,2}$	27.1964861	0.04231369	0.0269	0.3020	1.6966
22	$A_9^{22,2}$	28.8334329	0.04302673	0.0199	0.3348	1.7332
23	$A_9^{23,2}$	30.4963841	0.04371136	0.0281	0.3518	1.7708
24	$A_{10}^{24,2}$	32.1842434	0.04436768	0.0216	0.3865	1.8075
25	$A_{10}^{25,2}$	33.8958472	0.04499335	0.0295	0.4037	1.8464
26	$A_{11}^{26,2}$	35.6300014	0.04558788	0.0229	0.4392	1.8832
27	$B_{11,21}^{27,3}$	37.3854842	0.04615045	0.0304	0.4565	1.9222
28	$A_{12}^{28,2}$	39.1610622	0.04668075	0.0239	0.4928	1.9592
29	$B_{12,22}^{29,3}$	40.9555027	0.04717878	0.0307	0.5096	1.9978
30	$A_{13}^{30,2}$	42.7675849	0.04764484	0.0240	0.5463	2.0346
31	$A_{13}^{31,2}$	44.5961098	0.04807945	0.0304	0.5628	2.0727
32	$A_{14}^{32,2}$	46.4399090	0.04848337	0.0186	0.5922	2.1057
33	$A_{14}^{33,2}$	48.2978543	0.04885769	0.0243	0.6075	2.1383
34	$A_{14}^{34,2}$	50.1688626	0.04920355	0.0296	0.6222	2.1738
35	$B_{14,26}^{35,3}$	52.0518997	0.04952220	0.0338	0.6350	2.2073
36	$A_{15}^{36,2}$	53.9459843	0.04981500	0.0258	0.6710	2.2402
37	$B_{15,29}^{37,3}$	55.8501910	0.05008342	0.0307	0.6849	2.2740
38	$B_{15,29}^{38,3}$	57.7636509	0.05032891	0.0342	0.6960	2.3043
39	$B_{15,28}^{39,3}$	59.6855524	0.05055296	0.0366	0.7053	2.3329

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
40	$A_{15}^{40,3}$	61.6151397	0.05075705	0.0382	0.7129	2.3620
41	$A_{15}^{41,3}$	63.5517128	0.05094263	0.0397	0.7198	2.3888
42	$A_{15}^{42,3}$	65.4946250	0.05111110	0.0410	0.7262	2.4141
43	$A_{15}^{43,3}$	67.4432807	0.05126380	0.0422	0.7321	2.4379
44	$A_{15}^{44,3}$	69.3971334	0.05140202	0.0433	0.7376	2.4632
45	N_{16}^{45}	71.3556823	0.05152695	0.0395	0.7819	2.4876
46	N_{16}^{46}	73.3184698	0.05163973	0.0395	0.7852	2.5017
47	N_{16}^{47}	75.2850793	0.05174146	0.0395	0.7882	2.5146
48	N_{15}^{48}	77.2551336	0.05183320	0.0458	0.7534	2.5413
49	N_{15}^{49}	79.2282862	0.05191569	0.0458	0.7557	2.5502
50	N_{15}^{50}	81.2042263	0.05198993	0.0458	0.7577	2.5583
51	N_{15}^{51}	83.1826716	0.05205667	0.0458	0.7595	2.5657
52	N_{15}^{52}	85.1633672	0.05211663	0.0458	0.7611	2.5723
53	N_{14}^{53}	87.1460856	0.05217061	0.0492	0.7186	2.6042
54	N_{14}^{54}	89.1306147	0.05221882	0.0492	0.7197	2.6070
55	N_{14}^{55}	91.1167691	0.05226214	0.0492	0.7208	2.6095
56	N_{14}^{56}	93.1043803	0.05230098	0.0492	0.7217	2.6118
57	N_{14}^{57}	95.0932973	0.05233578	0.0492	0.7226	2.6138
58	N_{14}^{58}	97.0833838	0.05236696	0.0492	0.7234	2.6156
$s \geqslant 59$	D^s					

$$\sigma(19) = 0.004780 \quad (s = 50,\ t = 13,\ w = 28),$$
$$\tau(19) = 0.006252 \quad (s = 18), \quad G(19) \leqslant 134 \quad (v = 58).$$

22. Permissible exponents for twentieth powers

Following the computational procedure outlined in §8, we obtain the permissible exponents recorded in the table below. The exponents λ_s listed in the table improve on those previously available for $s \geqslant 3$. We follow a similar path to that taken in previous sections. Note first that for $3 \leqslant s \leqslant 23$, the condition (δ) of Theorem 3.4 is satisfied. Since one of the conditions (i) and (iii) of Theorem 3.4 is also satisfied in this range of s, we deduce that one may take $\tau_{j,l} = 0$ $(l = 1, 2)$ for $3 \leqslant s \leqslant 23$. When $10 \leqslant j \leqslant 14$ and $24 \leqslant s \leqslant 33$, meanwhile, we must resort to Theorem 3.4(II)(1) and (2). Here we note that condition (iii) is satisfied for $24 \leqslant s \leqslant 30$, and that (II)(2) applies for $s = 31, 32, 33$, and thus the estimate (4.1) holds for $10 \leqslant j \leqslant 14$ and $l = 2, 3$ with $\chi_{j,l} = \frac{1}{3}$ and $\tau_{j,l} = 1$. Note here that when $s = 31, 32, 33$, the relevant value of σ is so small that the condition $\phi_1 + \cdots + \phi_j \geqslant \frac{1}{3}(1 - \sigma)^{-1}$ is satisfied transparently.

In order to discuss permissible exponents for $34 \leqslant s \leqslant 47$, we apply case (Ic) of Theorem 3.4. In the notation of the latter theorem, we find from §23 that when $j = 15$ and $l = 2$, one has $J = 2$, and hence one may take

$$\delta_7 = 0.0332148, \quad \delta_8 = 0.0661378 \quad \text{and} \quad \delta_{16} = 0.6484502,$$

whence we may take

$$\tau_{15,l} = \tfrac{1}{15}(\delta_7 + \delta_8) < 0.00663 \quad \text{whenever} \quad \phi_1 \leqslant 0.048742 \quad (l = 2).$$

Further, when $j = 15, 16$ and $18 \leqslant j + l \leqslant 19$, one has $J = 1$, and hence one may take

$$\delta_7 = 0.1411109, \quad \delta_8 = 0.2285099, \quad \delta_{15} = 1.2643687, \quad \delta_{16} = 1.4713863.$$

Thus we deduce that one may take

$$\tau_{15,l} = \tfrac{1}{15}(\delta_7 + \delta_8) < 0.02465 \quad \text{whenever} \quad \phi_1 \leqslant 0.047586 \quad (l = 3),$$

and otherwise

$$\tau_{15,l} = \tfrac{1}{15}\delta_{15} < 0.08430 \quad (l = 2, 3),$$

and that one may take

$$\tau_{16,l} = \tfrac{1}{16}\delta_{16} < 0.09197 \quad (l = 2, 3).$$

As in the previous cases, our computations for $s \geqslant 48$ depend on first obtaining preliminary estimates by applying the process M_{14}^s for $48 \leqslant s \leqslant 61$ (noting (5.27) and checking (D_1) or (D_2)), and D^s for $s \geqslant 62$. In this way we obtain the preliminary permissible exponents

$$\lambda_{48} = 76.356, \quad \lambda_{49} = 78.320, \quad \lambda_{50} = 80.289, \quad \lambda_{51} = 82.260, \quad \lambda_{52} = 84.234,$$

$$\lambda_{53} = 86.211, \quad \lambda_{54} = 88.190, \quad \lambda_{55} = 90.171, \quad \lambda_{56} = 92.154, \quad \lambda_{57} = 94.138,$$

$$\lambda_{58} = 96.124, \quad \lambda_{59} = 98.112, \quad \lambda_{60} = 100.101, \quad \lambda_{61} = 102.091,$$

and by virtue of the preliminary exponent

$$\sigma(20) = 0.00450 \quad (s = 54,\ t = 14,\ w = 29),$$

we have also

$$\lambda_s = \max\{2s - 20,\ 102.091 + 2(s - 61)(1 - 0.00450)\}$$

for $s > 61$. Equipped with these preliminary bounds, we refine our procedure as indicated in the table. One may computationally check the validity of the appropriate case of Lemma 5.7 as follows.

(a) $s = 48, 49$. With process N_{17}^s, one finds that Lemma 5.7(I') holds with $u = s - 18$, by virtue of condition (A_1).

(b) $50 \leqslant s \leqslant 56$. With process N_{16}^s, one finds that Lemma 5.7(I') holds with $u = s - 17$, by virtue of condition (A_1).

(c) $57 \leqslant s \leqslant 62$. With process N_{15}^s, one finds that Lemma 5.7(I') holds with $u = s - 16$, by virtue of condition (A_1).

(e) $s \geqslant 63$. One finds that process D^s applies.

Table of permissible exponents for k = 20

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0035377	0.00176883			
4	$B_{1,6}^{4,2}$	4.0372112	0.01123775			
5	$B_{1,6}^{5,2}$	5.1214726	0.02126317			
6	$A_{1}^{6,2}$	6.2457501	0.02547440			
7	$B_{2,9}^{7,2}$	7.4042791	0.02754990	0.0141	0.0417	
8	$A_{2}^{8,2}$	8.5946271	0.02885933	0.0215	0.0504	
9	$A_{2}^{9,2}$	9.8176862	0.03012126	0.0263	0.0564	
10	$B_{3,10}^{10,2}$	11.0720258	0.03108406	0.0183	0.0781	1.0176
11	$A_{3}^{11,2}$	12.3569582	0.03191458	0.0241	0.0862	1.1705
12	$B_{4,12}^{12,2}$	13.6733507	0.03281045	0.0162	0.1093	1.2987
13	$A_{4}^{13,2}$	15.0199527	0.03356385	0.0225	0.1189	1.3522
14	$B_{5,13}^{14,2}$	16.3974554	0.03438079	0.0150	0.1436	1.4113
15	$A_{5}^{15,2}$	17.8050250	0.03512761	0.0221	0.1551	1.4486
16	$B_{6,14}^{16,2}$	19.2428091	0.03589874	0.0148	0.1814	1.4887
17	$A_{6}^{17,2}$	20.7101660	0.03663479	0.0221	0.1941	1.5221
18	$A_{6}^{18,2}$	22.2068593	0.03737393	0.0293	0.2069	1.5565
19	$A_{7}^{19,2}$	23.7322237	0.03808882	0.0225	0.2356	1.5891
20	$A_{8}^{20,2}$	25.2857385	0.03879475	0.0159	0.2652	1.6243
21	$A_{8}^{21,2}$	26.8666319	0.03947826	0.0234	0.2799	1.6577
22	$A_{9}^{22,2}$	28.4741610	0.04014501	0.0169	0.3107	1.6935
23	$A_{9}^{23,2}$	30.1074344	0.04078835	0.0244	0.3260	1.7277
24	$A_{10}^{24,2}$	31.7655438	0.04140990	0.0182	0.3583	1.7639
25	$A_{10}^{25,2}$	33.4474901	0.04200611	0.0258	0.3745	1.7992
26	$A_{11}^{26,2}$	35.1522491	0.04257718	0.0197	0.4079	1.8352
27	$B_{11,21}^{27,3}$	36.8787481	0.04312142	0.0271	0.4245	1.8717
28	$A_{12}^{28,2}$	38.6258968	0.04363868	0.0210	0.4587	1.9073
29	$B_{12,22}^{29,3}$	40.3925867	0.04412832	0.0279	0.4751	1.9443
30	$A_{13}^{30,2}$	42.1777082	0.04459040	0.0216	0.5097	1.9795
31	$A_{13}^{31,2}$	43.9801557	0.04502494	0.0282	0.5259	2.0165
32	$A_{14}^{32,2}$	45.7988398	0.04543236	0.0216	0.5606	2.0515
33	$B_{14,25}^{33,3}$	47.6326920	0.04581314	0.0276	0.5763	2.0876
34	$A_{15}^{34,2}$	49.4806744	0.04616803	0.0163	0.6042	2.1205
35	$A_{15}^{35,2}$	51.3417836	0.04649788	0.0215	0.6187	2.1507
36	$A_{15}^{36,2}$	53.2150565	0.04680367	0.0264	0.6325	2.1835
37	$B_{15,28}^{37,3}$	55.0995728	0.04708646	0.0308	0.6453	2.2165
38	$B_{15,28}^{38,3}$	56.9944581	0.04734736	0.0339	0.6559	2.2467
39	$A_{16}^{39,2}$	58.8988854	0.04758756	0.0276	0.6924	2.2796

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
40	$B_{16,30}^{40,3}$	60.8120763	0.04780826	0.0312	0.7037	2.3097
41	$B_{16,30}^{41,3}$	62.7333008	0.04801065	0.0337	0.7130	2.3370
42	$A_{16}^{42,3}$	64.6618772	0.04819593	0.0356	0.7208	2.3636
43	$A_{16}^{43,3}$	66.5971707	0.04836528	0.0369	0.7276	2.3899
44	$A_{16}^{44,3}$	68.5385926	0.04851983	0.0381	0.7338	2.4148
45	$A_{16}^{45,3}$	70.4855980	0.04866069	0.0393	0.7397	2.4384
46	$A_{16}^{46,3}$	72.4376841	0.04878890	0.0404	0.7451	2.4606
47	$A_{16}^{47,3}$	74.3943883	0.04890547	0.0414	0.7502	2.4837
48	N_{17}^{48}	76.3552854	0.04901133	0.0375	0.7919	2.5047
49	N_{17}^{49}	78.3199857	0.04910737	0.0375	0.7949	2.5178
50	N_{16}^{50}	80.2881335	0.04919448	0.0435	0.7623	2.5464
51	N_{16}^{51}	82.2594024	0.04927331	0.0435	0.7647	2.5556
52	N_{16}^{52}	84.2334960	0.04934469	0.0435	0.7668	2.5640
53	N_{16}^{53}	86.2101439	0.04940924	0.0435	0.7687	2.5717
54	N_{16}^{54}	88.1891004	0.04946760	0.0435	0.7704	2.5787
55	N_{16}^{55}	90.1701423	0.04952031	0.0435	0.7720	2.5851
56	N_{16}^{56}	92.1530669	0.04956792	0.0435	0.7735	2.5909
57	N_{15}^{57}	94.1376919	0.04961094	0.0468	0.7328	2.6199
58	N_{15}^{58}	96.1238487	0.04964967	0.0468	0.7338	2.6224
59	N_{15}^{59}	98.1113875	0.04968462	0.0468	0.7348	2.6246
60	N_{15}^{60}	100.1001722	0.04971613	0.0468	0.7356	2.6266
61	N_{15}^{61}	102.0900795	0.04974452	0.0468	0.7364	2.6284
62	N_{15}^{62}	104.0809983	0.04977011	0.0468	0.7371	2.6301
$s\geqslant 63$	D^s					

$$\sigma(20) = 0.004501 \quad (s = 54,\ t = 14,\ w = 29),$$

$$\tau(20) = 0.005915 \quad (s = 19), \quad G(20) \leqslant 142 \quad (v = 62).$$

23. Auxiliary permissible exponents

We collect together in this section the permissible exponents for larger k required in our calculation of the exponents in §§13–22. We do not work hard here to establish the sharpest such exponents, since good approximations will suffice.

Table of permissible exponents for k = 22

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0025439	0.00127191			
4	$B_{1,6}^{4,2}$	4.0292912	0.00892335			
5	$B_{1,6}^{5,2}$	5.1025520	0.01845031			
6	$A_1^{6,2}$	6.2121725	0.02238321			
7	$B_{2,9}^{7,2}$	7.3545751	0.02460381	0.0104	0.0350	
8	$B_{2,9}^{8,2}$	8.5257736	0.02576186	0.0183	0.0440	
9	$A_2^{9,2}$	9.7256814	0.02674629	0.0225	0.0492	
10	$B_{3,10}^{10,2}$	10.9550112	0.02771586	0.0146	0.0678	0.9494

Table of permissible exponents for k = 24

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0018708	0.00093536			
4	$B_{1,6}^{4,2}$	4.0234059	0.00718285			
5	$B_{1,7}^{5,2}$	5.0866022	0.01589207			
6	$A_1^{6,2}$	6.1846118	0.01994744			
7	$B_{2,10}^{7,2}$	7.3136219	0.02218427	0.0071	0.0293	
8	$B_{2,9}^{8,2}$	8.4689321	0.02322785	0.0154	0.0386	
9	$A_2^{9,2}$	9.6501924	0.02406834	0.0195	0.0436	
10	$B_{3,11}^{10,2}$	10.8586937	0.02497080	0.0108	0.0586	0.8645
11	$B_{3,11}^{11,2}$	12.0918680	0.02550777	0.0167	0.0662	1.0427

Table of permissible exponents for k = 26

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^j \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0014072	0.00070358			
4	$B_{1,6}^{4,2}$	4.0190100	0.00587034			
5	$B_{1,7}^{5,2}$	5.0738636	0.01377891			
6	$B_{1,7}^{6,2}$	6.1622494	0.01794221			
7	$B_{2,10}^{7,2}$	7.2802564	0.02021446	0.0045	0.0248	
8	$B_{2,10}^{8,2}$	8.4222375	0.02112895	0.0129	0.0341	
9	$A_2^{9,2}$	9.5880882	0.02188651	0.0171	0.0390	
10	$B_{3,11}^{10,2}$	10.7792591	0.02272621	0.0082	0.0516	0.8018
11	$B_{3,11}^{11,2}$	11.9929177	0.02317152	0.0141	0.0591	0.9909

Table of permissible exponents for k = 28

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0010768	0.00053837			
4	$B_{1,6}^{4,2}$	4.0156211	0.00484986			
5	$B_{1,8}^{5,2}$	5.0633584	0.01198110			
6	$B_{1,7}^{6,2}$	6.1434849	0.01623098			
7	$B_{2,10}^{7,2}$	7.2523170	0.01858310	0.0026	0.0212	
8	$B_{2,10}^{8,2}$	8.3829073	0.01935336	0.0107	0.0301	
9	$B_{2,10}^{9,2}$	9.5356712	0.02005540	0.0151	0.0351	
10	$A_2^{10,2}$	10.7106189	0.02066883	0.0179	0.0385	
11	$B_{3,12}^{11,2}$	11.9079284	0.02124033	0.0115	0.0526	0.9281
12	$A_3^{12,2}$	13.1262695	0.02163492	0.0154	0.0576	1.0629

Table of permissible exponents for k = 30

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0008289	0.00041441			
4	$B_{1,6}^{4,2}$	4.0128519	0.00400878			
5	$B_{1,8}^{5,2}$	5.0536213	0.01022522			
6	$B_{1,7}^{6,2}$	6.1264298	0.01471957			
7	$A_1^{7,2}$	7.2247478	0.01673904			
8	$B_{2,11}^{8,2}$	8.3459599	0.01789043	0.0087	0.0266	
9	$B_{2,10}^{9,2}$	9.4875943	0.01850453	0.0134	0.0319	
10	$A_2^{10,2}$	10.6496179	0.01903383	0.0160	0.0350	
11	$B_{3,13}^{11,2}$	11.8328152	0.01959249	0.0094	0.0472	0.8726
12	$A_3^{12,2}$	13.0354619	0.01993146	0.0135	0.0523	1.0225
13	$B_{4,14}^{13,2}$	14.2585847	0.02034949	0.0061	0.0641	1.1865
14	$B_{4,14}^{14,2}$	15.5012985	0.02067161	0.0113	0.0708	1.2261

Table of permissible exponents for k = 32

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0006513	0.00032563			
4	$B_{1,6}^{4,2}$	4.0107492	0.00336671			
5	$B_{1,8}^{5,2}$	5.0460456	0.00884787			
6	$B_{1,8}^{6,2}$	6.1127790	0.01347074			
7	$A_1^{7,2}$	7.2031814	0.01535571			
8	$B_{2,11}^{8,2}$	8.3160843	0.01661114	0.0070	0.0236	
9	$B_{2,11}^{9,2}$	9.4480157	0.01716982	0.0118	0.0290	
10	$B_{2,11}^{10,2}$	10.5989372	0.01764754	0.0145	0.0321	
11	$A_2^{11,2}$	11.7694872	0.01814158	0.0166	0.0348	
12	$B_{3,13}^{12,2}$	12.9585035	0.01847574	0.0117	0.0477	0.9802
13	$A_3^{13,2}$	14.1659645	0.01878922	0.0145	0.0513	1.0906
14	$B_{4,15}^{14,2}$	15.3924025	0.01913448	0.0091	0.0641	1.2034
15	$A_4^{15,2}$	16.6372483	0.01942050	0.0127	0.0690	1.2378

Table of permissible exponents for k = 34

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0005147	0.00025732			
4	$B_{1,6}^{4,2}$	4.0090083	0.00283170			
5	$B_{1,8}^{5,2}$	5.0397401	0.00770030			
6	$B_{1,8}^{6,2}$	6.1004200	0.01223322			
7	$A_1^{7,2}$	7.1840767	0.01418011			
8	$B_{2,12}^{8,2}$	8.2896184	0.01548459	0.0054	0.0209	
9	$B_{2,11}^{9,2}$	9.4129882	0.01600048	0.0104	0.0264	
10	$B_{2,11}^{10,2}$	10.5541305	0.01643672	0.0131	0.0295	
11	$A_2^{11,2}$	11.7133896	0.01686019	0.0150	0.0319	
12	$B_{3,13}^{12,2}$	12.8905163	0.01721916	0.0101	0.0436	0.9376
13	$A_3^{13,2}$	14.0847742	0.01748578	0.0129	0.0471	1.0521
14	$A_3^{14,2}$	15.2966345	0.01778064	0.0154	0.0504	1.1577
15	$B_{4,15}^{15,2}$	16.5259780	0.01805376	0.0109	0.0632	1.2170

Table of permissible exponents for k = 36

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0004128	0.00020635			
4	$B_{1,6}^{4,2}$	4.0076365	0.00240826			
5	$B_{1,8}^{5,2}$	5.0346195	0.00675866			
6	$B_{1,8}^{6,2}$	6.0901459	0.01118271			
7	$A_1^{7,2}$	7.1679365	0.01316287			
8	$B_{2,12}^{8,2}$	8.2670449	0.01450637	0.0040	0.0185	
9	$B_{2,12}^{9,2}$	9.3828824	0.01497972	0.0091	0.0241	
10	$B_{2,11}^{10,2}$	10.5154507	0.01538430	0.0118	0.0272	
11	$A_2^{11,2}$	11.6648323	0.01575000	0.0137	0.0294	
12	$B_{3,14}^{12,2}$	12.8314116	0.01611772	0.0086	0.0398	0.8917
13	$A_3^{13,2}$	14.0140325	0.01635130	0.0115	0.0435	1.0180
14	$A_3^{14,2}$	15.2130601	0.01660506	0.0137	0.0464	1.1166
15	$B_{4,16}^{15,2}$	16.4286845	0.01686286	0.0093	0.0581	1.1986
16	$A_4^{16,2}$	17.6606068	0.01708916	0.0120	0.0618	1.2277

Table of permissible exponents for k = 38

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0003353	0.00016763			
4	$B_{1,6}^{4,2}$	4.0065387	0.00206802			
5	$B_{1,8}^{5,2}$	5.0304085	0.00597725			
6	$B_{1,9}^{6,2}$	6.0814901	0.01027884			
7	$A_1^{7,2}$	7.1541435	0.01227563			
8	$B_{2,12}^{8,2}$	8.2475913	0.01365028	0.0029	0.0166	
9	$B_{2,12}^{9,2}$	9.3566575	0.01406869	0.0079	0.0220	
10	$B_{2,11}^{10,2}$	10.4816593	0.01446221	0.0108	0.0253	
11	$A_2^{11,2}$	11.6223319	0.01477912	0.0125	0.0273	
12	$B_{3,14}^{12,2}$	12.7795262	0.01514737	0.0072	0.0366	0.8484
13	$B_{3,14}^{13,2}$	13.9517878	0.01535244	0.0102	0.0402	0.9829
14	$A_3^{14,2}$	15.1394461	0.01557562	0.0123	0.0429	1.0800
15	$A_3^{15,2}$	16.3428361	0.01581504	0.0144	0.0457	1.1800
16	$A_4^{16,2}$	17.5615789	0.01601671	0.0105	0.0571	1.2100

Table of permissible exponents for k = 40

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0002717	0.00013584			
4	$B_{1,6}^{4,2}$	4.0055702	0.00176633			
5	$B_{1,8}^{5,2}$	5.0266358	0.00527374			
6	$B_{1,9}^{6,2}$	6.0730909	0.00934078			
7	$A_1^{7,2}$	7.1411109	0.01147649			
8	$A_1^{8,2}$	8.2285099	0.01274245			
9	$B_{2,13}^{9,2}$	9.3316986	0.01327785	0.0069	0.0201	
10	$B_{2,12}^{10,2}$	10.4498752	0.01363320	0.0098	0.0235	
11	$A_2^{11,2}$	11.5828707	0.01392605	0.0115	0.0255	
12	$A_2^{12,2}$	12.7314053	0.01425870	0.0132	0.0274	
13	$B_{3,14}^{13,2}$	13.8944894	0.01447244	0.0091	0.0374	0.9509
14	$A_3^{14,2}$	15.0720426	0.01466715	0.0111	0.0399	1.0482
15	$A_3^{15,2}$	16.2643687	0.01487677	0.0130	0.0424	1.1421
16	$A_4^{16,2}$	17.4713863	0.01507158	0.0093	0.0531	1.1947
17	$A_5^{17,2}$	18.6931953	0.01526705	0.0056	0.0641	1.2358

Table of permissible exponents for k = 56

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000641	0.00003202			
4	$B_{1,6}^{4,2}$	4.0018248	0.00058692			
5	$B_{1,8}^{5,2}$	5.0107720	0.00223784			
6	$B_{1,10}^{6,2}$	6.0336021	0.00457587			
7	$B_{1,10}^{7,2}$	7.0749443	0.00692919			
8	$B_{1,10}^{8,2}$	8.1314920	0.00816567			
9	$A_1^{9,2}$	9.2015653	0.00890554			
10	$B_{2,15}^{10,2}$	10.2835673	0.00932007	0.0041	0.0134	
11	$B_{2,14}^{11,2}$	11.3759867	0.00951166	0.0063	0.0158	
12	$B_{2,14}^{12,2}$	12.4787878	0.00967630	0.0076	0.0173	
13	$A_2^{13,2}$	13.5920033	0.00982670	0.0086	0.0184	
14	$A_2^{14,2}$	14.7160440	0.00999685	0.0095	0.0195	
15	$A_3^{15,2}$	15.8501204	0.01009311	0.0067	0.0265	0.9447
16	$A_4^{16,2}$	16.9946507	0.01021425	0.0040	0.0337	1.1211

Table of permissible exponents for k = 60

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000471	0.00002352			
4	$B_{1,6}^{4,2}$	4.0014347	0.00046255			
5	$B_{1,8}^{5,2}$	5.0088563	0.00185607			
6	$B_{1,10}^{6,2}$	6.0283226	0.00390017			
7	$B_{1,11}^{7,2}$	7.0648348	0.00611424			
8	$B_{1,10}^{8,2}$	8.1162996	0.00742085			
9	$A_{1}^{9,2}$	9.1804196	0.00813325			
10	$B_{2,16}^{10,2}$	10.2565541	0.00863245	0.0031	0.0117	
11	$B_{2,15}^{11,2}$	11.3423160	0.00880201	0.0053	0.0142	
12	$B_{2,15}^{12,2}$	12.4377875	0.00895800	0.0068	0.0157	
13	$A_{2}^{13,2}$	13.5428594	0.00908753	0.0077	0.0167	
14	$A_{2}^{14,2}$	14.6578046	0.00922726	0.0085	0.0177	
15	$A_{3}^{15,2}$	15.7824957	0.00934563	0.0059	0.0242	0.9129
16	$A_{4}^{16,2}$	16.9168390	0.00944915	0.0033	0.0308	1.1104

Table of permissible exponents for k = 64

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000353	0.00001762			
4	$B_{1,6}^{4,2}$	4.0011451	0.00036996			
5	$B_{1,8}^{5,2}$	5.0073658	0.00155562			
6	$B_{1,10}^{6,2}$	6.0241263	0.00335705			
7	$B_{1,11}^{7,2}$	7.0565053	0.00541830			
8	$B_{1,11}^{8,2}$	8.1035626	0.00677718			
9	$A_{1}^{9,2}$	9.1626169	0.00747861			
10	$B_{2,16}^{10,2}$	10.2336743	0.00804055	0.0022	0.0103	
11	$B_{2,16}^{11,2}$	11.3136439	0.00818830	0.0045	0.0127	
12	$B_{2,15}^{12,2}$	12.4026992	0.00833355	0.0060	0.0143	
13	$B_{2,15}^{13,2}$	13.5007324	0.00845312	0.0069	0.0153	
14	$A_{2}^{14,2}$	14.6078413	0.00856921	0.0076	0.0162	
15	$B_{3,18}^{15,2}$	15.7242889	0.00869521	0.0047	0.0216	0.8471
16	$B_{3,18}^{16,2}$	16.8494116	0.00876473	0.0059	0.0232	0.9414
17	$B_{4,18}^{17,2}$	17.9835998	0.00885697	0.0031	0.0289	1.1088
18	$B_{5,18}^{18,2}$	19.1266047	0.00892866	0.0004	0.0348	1.1504

Table of permissible exponents for k = 68

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000267	0.00001331			
4	$B_{1,6}^{4,2}$	4.0009181	0.00029715			
5	$B_{1,8}^{5,2}$	5.0061304	0.00130338			
6	$B_{1,10}^{6,2}$	6.0205621	0.00288990			
7	$B_{1,12}^{7,2}$	7.0489837	0.00475323			
8	$B_{1,11}^{8,2}$	8.0918863	0.00617213			
9	$B_{1,11}^{9,2}$	9.1465073	0.00690696			
10	$A_1^{10,2}$	10.2122415	0.00742466			
11	$B_{2,17}^{11,2}$	11.2872180	0.00766025	0.0037	0.0114	
12	$B_{2,16}^{12,2}$	12.3706630	0.00778929	0.0053	0.0130	
13	$B_{2,15}^{13,2}$	13.4625637	0.00790250	0.0062	0.0142	
14	$A_2^{14,2}$	14.5628858	0.00800180	0.0069	0.0149	
15	$A_2^{15,2}$	15.6719470	0.00811643	0.0076	0.0158	
16	$B_{3,19}^{16,2}$	16.7893310	0.00819260	0.0051	0.0212	0.9024
17	$B_{3,19}^{17,2}$	17.9149233	0.00825686	0.0061	0.0224	0.9791
18	$A_3^{18,2}$	19.0488059	0.00832341	0.0069	0.0234	1.0441
19	$A_4^{19,2}$	20.1911221	0.00839565	0.0047	0.0293	1.1259
20	$A_5^{20,2}$	21.3417807	0.00845975	0.0024	0.0352	1.1581

Table of permissible exponents for k = 72

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000205	0.00001021			
4	$B_{1,6}^{4,2}$	4.0007451	0.00024154			
5	$B_{1,8}^{5,2}$	5.0051551	0.00110272			
6	$B_{1,10}^{6,2}$	6.0176775	0.00250706			
7	$B_{1,12}^{7,2}$	7.0427686	0.00419421			
8	$B_{1,12}^{8,2}$	8.0820772	0.00565004			
9	$B_{1,11}^{9,2}$	9.1327314	0.00639741			
10	$A_{1}^{10,2}$	10.1937853	0.00688532			
11	$B_{2,18}^{11,2}$	11.2643373	0.00719463	0.0031	0.0102	
12	$B_{2,17}^{12,2}$	12.3428217	0.00731063	0.0046	0.0119	
13	$B_{2,16}^{13,2}$	13.4292736	0.00741620	0.0056	0.0131	
14	$A_{2}^{14,2}$	14.5236334	0.00750632	0.0063	0.0138	
15	$A_{2}^{15,2}$	15.6261004	0.00760346	0.0070	0.0146	
16	$B_{3,20}^{16,2}$	16.7366445	0.00769062	0.0044	0.0195	0.8628
17	$B_{3,19}^{17,2}$	17.8548813	0.00774646	0.0054	0.0207	0.9486
18	$A_{3}^{18,2}$	18.9808873	0.00780459	0.0062	0.0216	1.0116
19	$A_{3}^{19,2}$	20.1147519	0.00786555	0.0070	0.0226	1.0785
20	$B_{4,22}^{20,2}$	21.2564959	0.00792520	0.0048	0.0280	1.1276
21	$B_{5,22}^{21,2}$	22.4061171	0.00798257	0.0024	0.0334	1.1564
22	$A_{6}^{22,2}$	23.5636041	0.00803757	0.0006	0.0394	1.1688

Table of permissible exponents for k = 76

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j} \phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000159	0.00000793			
4	$B_{1,6}^{4,2}$	4.0006104	0.00019818			
5	$B_{1,8}^{5,2}$	5.0043590	0.00093729			
6	$B_{1,10}^{6,2}$	6.0152939	0.00218890			
7	$B_{1,12}^{7,2}$	7.0375713	0.00372240			
8	$B_{1,12}^{8,2}$	8.0734591	0.00515450			
9	$B_{1,11}^{9,2}$	9.1206671	0.00595570			
10	$A_1^{10,2}$	10.1776325	0.00641551			
11	$B_{2,18}^{11,2}$	11.2442251	0.00677969	0.0024	0.0092	
12	$B_{2,17}^{12,2}$	12.3182827	0.00688538	0.0040	0.0109	
13	$B_{2,17}^{13,2}$	13.3998892	0.00698584	0.0051	0.0121	
14	$A_2^{14,2}$	14.4889692	0.00706978	0.0058	0.0129	
15	$A_2^{15,2}$	15.5856100	0.00715274	0.0064	0.0135	
16	$B_{3,21}^{16,2}$	16.6900569	0.00724602	0.0037	0.0179	0.8224
17	$B_{3,20}^{17,2}$	17.8017421	0.00729495	0.0048	0.0192	0.9160
18	$A_3^{18,2}$	18.9207482	0.00734685	0.0056	0.0201	0.9828
19	$A_3^{19,2}$	20.0471373	0.00740016	0.0063	0.0209	1.0454
20	$A_3^{20,2}$	21.1809902	0.00745580	0.0070	0.0219	1.1119
21	$B_{4,23}^{21,2}$	22.3222235	0.00750483	0.0050	0.0271	1.1305
22	$B_{5,24}^{22,2}$	23.4709154	0.00755634	0.0026	0.0319	1.1551
23	$A_5^{23,2}$	24.6270386	0.00760498	0.0038	0.0335	1.1634
24	$A_6^{24,2}$	25.7906446	0.00765482	0.0018	0.0389	1.1744

Table of permissible exponents for k = 80

s	Process	λ_s	ϕ_1	ϕ_j	$\sum_{i=1}^{j}\phi_i$	ϕ_s^*
3	$B_{1,4}^{3,2}$	3.0000125	0.00000624			
4	$B_{1,6}^{4,2}$	4.0005051	0.00016419			
5	$B_{1,8}^{5,2}$	5.0037183	0.00080341			
6	$B_{1,10}^{6,2}$	6.0133258	0.00192294			
7	$B_{1,12}^{7,2}$	7.0332148	0.00332221			
8	$B_{1,12}^{8,2}$	8.0661378	0.00472571			
9	$B_{1,12}^{9,2}$	9.1101284	0.00554468			
10	$A_1^{10,2}$	10.1634946	0.00600304			
11	$B_{2,18}^{11,2}$	11.2265639	0.00641176	0.0019	0.0083	
12	$B_{2,18}^{12,2}$	12.2966473	0.00650521	0.0035	0.0100	
13	$B_{2,17}^{13,2}$	13.3738933	0.00660034	0.0046	0.0112	
14	$B_{2,17}^{14,2}$	14.4582437	0.00668064	0.0053	0.0120	
15	$A_2^{15,2}$	15.5496954	0.00675332	0.0059	0.0126	
16	$A_2^{16,2}$	16.6484502	0.00683410	0.0064	0.0133	
17	$B_{3,21}^{17,2}$	17.7542883	0.00689430	0.0042	0.0178	0.8823
18	$A_3^{18,2}$	18.8670322	0.00693992	0.0050	0.0187	0.9570
19	$A_3^{19,2}$	19.9867390	0.00698693	0.0057	0.0195	1.0160
20	$A_3^{20,2}$	21.1134792	0.00703595	0.0063	0.0203	1.0785
21	$B_{4,24}^{21,2}$	22.2472590	0.00708335	0.0044	0.0252	1.1204
22	$B_{5,24}^{22,2}$	23.3880855	0.00712947	0.0020	0.0297	1.1467
23	$A_5^{23,2}$	24.5359375	0.00717314	0.0032	0.0312	1.1545
24	$A_6^{24,2}$	25.6908623	0.00721788	0.0013	0.0363	1.1651

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